MATH 127
MIDTERM II SOLUTIONS
April 3, 2009

1. (30 pts) Compute the following integrals. The third one is improper.

(a) \[ \int \sec^2 \theta \, d\theta = \tan \theta + C. \]

(b) \[ \int \cos^3(x) \sin^4(x) \, dx = \int \cos^2(x) \sin^4(x) \cos(x) \, dx = \int (1 - \sin^2(x)) \sin^4(x) \cos(x) \, dx. \]

Let \( u = \sin(x), \) \( du = \cos(x) \, dx. \) Then the above is

\[ = \int (1 - u^2)u^4 \, du = \int (u^4 - u^6) \, du = \frac{u^5}{5} - \frac{u^7}{7} + C \]

\[ = \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C. \]

(c) \[ \int_0^2 \frac{1}{x^2} \, dx = \lim_{R \to 0^+} \int_R^2 x^{-2} \, dx = \lim_{R \to 0^+} -x^{-1} \bigg|_R^2 = \lim_{R \to 0^+} \left( -\frac{1}{2} - \frac{1}{R} \right) = \lim_{R \to 0^+} \frac{1}{R} - \frac{1}{2} = \infty. \]

(The integral diverges.)

2. (20 pts) Griswald puts a frozen hot dog into a pot of boiling water. The temperature of the water is 100\(^\circ\)C. The temperature of the hot dog at time \( t = 0 \) is 0\(^\circ\)C. After 2 minutes, the hot dog is 5\(^\circ\)C.

(a) By Newton’s law of cooling, the rate of change of the temperature of the hot dog is proportional to the difference in temperature between the water and the hot dog. Write this as an IVP. We let \( y = y(t) \) denote the temperature at time \( t \).

\[ \frac{dy}{dt} = k(100 - y), \quad y(0) = 0, \quad y(2) = 5. \]

(b) Solve the IVP to obtain the temperature of the hot dog at time \( t \).

\[ \frac{1}{100 - y} \, dy = k \, dt \quad \Rightarrow \quad \int \frac{1}{100 - y} \, dy = kt + C. \]

In the \( y \)-integral, we can let \( u = 100 - y, \) \( du = -dy. \) We get:

\[ -\ln |100 - y| = kt + C \quad \Rightarrow \quad \ln(100 - y) = -kt - C, \]
which gives

\[ 100 - y = Ae^{-kt}, \]

where \( A \) is a positive constant (equal to \( e^{-C} \)). Setting \( t = 0 \) and \( y = 0 \) (the initial value) immediately gives \( A = 100 \). Thus

\[ 100 - y = 100e^{-kt}. \]

Set \( t = 2 \) and \( y = 5 \):

\[ 95 = 100e^{-2k} \implies .95 = e^{-2k} \implies \ln(.95) = -2k \implies k = -\frac{\ln(.95)}{2}. \]

Putting this all together (note the minus sign from \( k \) cancels in \( e^{-kt} \)), we find:

\[ y(t) = 100 - 100e^{t\ln(.95)/2}. \]

(c) At what time will the hot dog be 60°C?

We set:

\[ 60 = y(t) = 100 - 100e^{t\ln(.95)/2}. \]

Now solve for \( t \):

\[ 100e^{t\ln(.95)/2} = 40 \implies e^{t\ln(.95)/2} = .4 \]

\[ t\frac{\ln(.95)}{2} = \ln(.4) \implies t = \frac{2\ln(.4)}{\ln(.95)} \approx 35.7 \text{ minutes}. \]

3. (18 pts) Compute the integral

\[ \int \frac{x^2}{(\sqrt{x^2 - 9})^5} dx. \]

Let \( x = 3 \sec \theta \), so \( dx = 3 \sec \theta \tan \theta \). Then

\[ \sqrt{x^2 - 9} = \sqrt{9 \sec^2 \theta - 9} = 3 \sqrt{\sec^2 \theta - 1} = 3 \sqrt{\tan^2 \theta} = 3 \tan \theta. \]

The integral thus becomes

\[ \int \frac{3^2 \sec^2 \theta}{(3 \tan \theta)^5} 3 \sec \theta \tan \theta \, d\theta = \frac{3^3}{3^5} \int \frac{\sec^3 \theta}{\tan^4 \theta} \, d\theta. \]

There’s no obvious way to integrate this directly. So we convert to sines and cosines and hope for the best:

\[ = \frac{1}{9} \int \frac{1}{\cos^3 \theta \sin^4 \theta} \, d\theta = \frac{1}{9} \int \frac{\cos \theta}{\sin^4 \theta} \, d\theta. \]
Let \( u = \sin \theta \), \( du = \cos \theta \, d\theta \). We get

\[
\frac{1}{9} \int u^{-4} du = \frac{1}{9} \cdot \frac{u^{-3}}{-3} = -\frac{1}{27 \sin^3 \theta}.
\]

To convert back to \( x \), note that \( \sec \theta = \frac{x}{3} = \frac{\text{hyp}}{\text{adj}} \), so we have:

\[
\theta \quad \sqrt{x^2 - 9} \\
3 \quad x
\]

Therefore \( \sin \theta = \frac{\sqrt{x^2 - 9}}{x} \), so the integral is

\[
= -\frac{1}{27} \left( \frac{x}{\sqrt{x^2 - 9}} \right)^3 + C.
\]

4. (18 pts) Use the method of partial fractions to compute:

\[
\int \frac{1}{x^2(x - 1)} dx.
\]

According the PF-decomposition, there exist constants \( A, B, C \) such that

\[
\frac{1}{x^2(x - 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x - 1}.
\]

Clearing denominators gives:

\[
1 = Ax(x - 1) + B(x - 1) + Cx^2.
\]

Setting \( x = 0 \) gives \( 1 = -B \), so \( B = -1 \). Setting \( x = 1 \) gives \( 1 = C \). The coefficient of \( x^2 \) on the left is 0. The coefficient of \( x^2 \) on the right is \( A + C = A + 1 \). Therefore \( A + 1 = 0 \), so \( A = -1 \). Thus

\[
\int \frac{1}{x^2(x - 1)} dx = \int -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x - 1} dx
\]

\[
= -\ln |x| + \frac{1}{x} + \ln |x - 1| + C.
\]
5. **(14 pts)** The vertical ends of a water trough are isosceles triangles of base 4 m and height 3 m. Find the hydrostatic force on one end if the trough is full. The density of water is 1000 kg/m$^3$.

![Diagram of a water trough with dimensions labeled]

The force of water on a thin slice of width $dy$ at height $y$ is

$$F_{\text{slice}} = \text{Pressure} \cdot \text{Area} = \rho g D(y) W(y) dy,$$

where $\rho$ is the density of the fluid, $g$ is gravitational acceleration, $D(y)$ is the depth of the fluid at $y$, and $W(y)$ is the width of the plate at $y$. We “add” these increments of force by integrating, so the total force on a submerged plate, extending from $y = a$ at the bottom to $y = b$ at the top is

$$F = \int_a^b \rho g D(y) W(y) dy.$$

In the picture, we have placed the origin at the bottom. The equation for the right edge of the trough is $y = \frac{3}{2}x$ (since slope $= \frac{\text{rise}}{\text{run}} = \frac{3}{2}$). The width of the trough at $y$ is twice the $x$-coordinate:

$$W(y) = 2x = 2 \cdot \frac{2}{3}y = \frac{4}{3}y.$$

The depth at $y$ is $D(y) = 3 - y$. Therefore

$$F = (1000)(9.8) \int_0^3 (3 - y) \frac{4}{3}y dy = (9800) \frac{4}{3} \int_0^3 (3y - y^2) dy$$

$$= \frac{4}{3}(9800) \left[ \frac{3y^2}{2} - \frac{y^3}{3} \right]_0^3 = \frac{4}{3}(9800)(\frac{27}{2} - 9)$$

$$= 9800(18 - 12) = (9800)(6) \text{ Newtons.}$$