1. (20 pts) Consider the following plate, submerged vertically under water.

(a) Write down an integral representing the length of the curve. Do not evaluate the integral.

\[ L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + (2x)^2} \, dx. \]

(b) Which of the following integrals integral represents the total force \( F \) (in Newtons) on the plate? (Distance above is measured in meters. You do not need to show any work.)

A. \( \int_{-2}^{0} (1000)(9.8)(-y) 2\sqrt{y + 2} \, dy \)

B. \( \int_{-\sqrt{2}}^{\sqrt{2}} (1000)(9.8)2x(x^2 - 2) \, dx \)

C. \( \int_{-2}^{0} (1000)(9.8)y\sqrt{y + 2} \, dy \)

D. \( -(1000)(9.8)2 \int_{-\sqrt{2}}^{\sqrt{2}} (x^2 - 2) \, dx \)
The force is

\[ F = \int_a^b \rho d(y) W(y) dy, \]

where \( d(y) \) is the depth, and \( W(y) \) is the width of the plate at \( y \). Here \( d(y) = (-y) \) and \( W(y) = 2x = 2\sqrt{y+2} \). Therefore

\[ F = \int_{-2}^0 (1000)(9.8)(-y)2\sqrt{y+2} dy. \]

2. (20 pts) Compute \( \int \frac{1}{\sqrt{x^2 + 4}} \ dx \)

Let \( x = 2 \tan \theta \) for \(-\pi/2 \leq \theta \leq \pi/2\). Then \( dx = 2 \sec^2 \theta \ d\theta \), and

\[ \sqrt{x^2 + 4} = \sqrt{4 \tan^2 + 4} = \sqrt{4 \sec^2 \theta} = 2 \sec \theta. \]

Now

\[ \int \frac{1}{\sqrt{x^2 + 4}} \ dx = \int \frac{2 \sec^2 \theta}{2 \sec \theta} \ d\theta = \int \sec \theta \ d\theta = \ln |\sec \theta + \tan \theta| + C. \]

To go back to the original variable, we use the triangle:

\[ \tan \theta = \frac{x}{2} \]

which comes from our original definition: \( \tan \theta = \frac{x}{2} \).

Now we see that

\[ \int \frac{1}{\sqrt{x^2 + 4}} \ dx = \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C. \]

3. (22 pts)

(a) (12 pts) Compute the following improper integral. Give the value if convergent, or show that it diverges. For full credit, you must use the correct definition of the integral. (In particular, your answer must involve taking a limit.)

\[ \int_{2}^{\infty} e^{-5x} \ dx = \lim_{R \to \infty} \int_{2}^{R} e^{-5x} \ dx = \lim_{R \to \infty} \left[ \frac{e^{-5x}}{-5} \right]_{2}^{R} = \lim_{R \to \infty} \left( \frac{e^{-5R}}{-5} - \frac{e^{-10}}{-5} \right) = \frac{e^{-10}}{5}. \]

(b) (10 pts) Compute:

\[ \int \sin^2(x) \ dx = \int \frac{1 - \cos(2x)}{2} \ dx \]

\[ = \frac{1}{2} \int (1 - \cos(2x)) \ dx = \frac{1}{2} \left( x - \frac{\sin(2x)}{2} \right) + C. \]
4. (20 pts) Use Partial Fractions to compute:

\[ \int \frac{y + 2}{y^2 + y} \, dy \]

First note that the degree of the top is less than the degree of the bottom, so we do not need to do long division. Next, factor the denominator: \( y^2 + y = y(y + 1) \). Therefore

\[ \frac{y + 2}{y(y + 1)} = \frac{A}{y} + \frac{B}{y + 1}. \]

Cross-multiplying,

\[ y + 2 = A(y + 1) + By. \]

Setting \( y = 0 \) gives \( A = 2 \). Setting \( y = -1 \) gives \( 1 = -B \). Therefore

\[ \int \frac{y + 2}{y^2 + y} \, dy = \int \left( \frac{2}{y} - \frac{1}{y + 1} \right) \, dy \]

\[ = 2 \ln |y| - \ln |y + 1| + C = \ln |y^2| - \ln |y + 1| + C = \ln \left| \frac{y^2}{y + 1} \right| + C. \]

5. (18 pts) Find a solution \( y = y(x) \) to the following IVP:

\[ \frac{dy}{dx} = \frac{\cos x}{2y} \quad (y(0) = 4). \]

Separating variables, we have

\[ 2y \, dy = \cos x \, dx. \]

Integrating,

\[ \int 2y \, dy = \int \cos x \, dx \implies y^2 = \sin(x) + C. \]

Setting \( x = 0 \) and \( y = 4 \) gives \( 16 = C \). Therefore the desired solution is

\[ y = \sqrt{\sin(x) + 16}. \]

(Note: We took the positive square root since we are given that \( y \) is positive when \( x = 0 \).)