1. (20 pts) Compute the following.

(a) \( \int_{0}^{4} \frac{1}{\sqrt{3x + 4}} \, dx \)

Let \( u = 3x + 4 \). Then \( du = 3 \, dx \). We need to convert the limits: When \( x = 0, u = 4 \). When \( x = 4, u = 16 \). Thus the integral is equal to

\[
\int_{4}^{16} u^{-1/2} \frac{du}{3} = \frac{1}{3} \left[ u^{1/2} \right]_{4}^{16} = \frac{2}{3} (\sqrt{16} - \sqrt{4}) = \frac{2}{3} (4 - 2) = \frac{4}{3}.
\]

(b) \( \int \arctan(x) \, dx \)

Here we use IBP. Let:

\[
u = \arctan x \quad dv = dx
\]
\[
 du = \frac{1}{1 + x^2} \, dx \quad v = x.
\]

Then

\[
\int \arctan(x) \, dx = x \arctan(x) - \int \frac{x}{1 + x^2} \, dx
\]

To solve the last integral, let \( u = 1 + x^2, du = 2x \, dx \). Then the above is

\[
x \arctan(x) - \frac{1}{2} \int u^{-1} \, du = x \arctan(x) - \frac{1}{2} \ln |1 + x^2| + C.
\]

2. (20 pts)

(a) Compute \( \int e^{\sin(x)} \cos(x) \, dx \)

Let \( u = \sin(x), \, du = \cos(x) \, dx \). Then the integral is equal to

\[
\int e^u \, du = e^u + C = e^{\sin(x)} + C.
\]
(b) What is the average value of 
\[ y = x \cos x \]

on the interval \([-\pi, \pi]\)?

We know that the average value of \(f(x)\) on \([a, b]\) is \( \frac{1}{b-a} \int_a^b f(x)dx \).

Easy Solution: The function \(f(x) = x \cos(x)\) is an odd function, i.e. \( f(-x) = -f(x) \). Therefore its integral over any symmetric interval (e.g. \([-\pi, \pi]\)) vanishes. So

\[
 f_{\text{ave}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cos(x)dx = 0.
\]

Hard Solution: If you fail to notice the symmetry, then you have to compute the integral. Use IBP:

\[
 u = x \quad dv = \cos(x)dx \\
 du = dx \quad v = \sin(x).
\]

\[
 f_{\text{ave}} = \frac{1}{2\pi} \left( x \sin(x)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin(x)dx \right).
\]

Once again, the integral of \(\sin(x)\) vanishes by symmetry since \(\sin(x)\) is odd. But failing to notice this, we continue:

\[
 f_{\text{ave}} = \frac{1}{2\pi} [\pi \sin(\pi) - (-\pi) \sin(-\pi)] - \frac{1}{2\pi} [-\cos(\pi) - (-\cos(-\pi))]
\]

\[
 = 0 - \frac{1}{2\pi}[1 - 1] = 0.
\]

3. (20 pts)

(a) Use Simpson’s rule with \(n = 4\) to approximate \( \ln 5 = \int_1^5 \frac{1}{x}dx \).

Recall the formula

\[
 S_n = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 4f(x_{n-1}) + f(x_n)].
\]

In our case, \(f(x) = 1/x\), and when \(n = 4\), \(\Delta x = \frac{5-1}{4} = 1\). Thus

\[
 \ln 5 \approx \frac{1}{3} [f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)] = \frac{1}{3} [1 + \frac{4}{2} + \frac{2}{3} + \frac{4}{4} + \frac{1}{5}].
\]
(b) Find a bound for the error of your estimate in part (a). (Recall that $ES_n \leq \frac{K_4(b-a)^5}{180n^4}$.)

Here $K_4$ is defined by $|f^{(4)}(x)| \leq K_4$ for all $x \in [1, 5]$. In our case, we compute easily that $f^{(4)}(x) = \frac{24}{x^5}$. This is largest when $x$ is as small as possible, i.e. when $x = 1$. So we can take $K_4 = 24$. Therefore the magnitude of the error in part (a) is

$$ES_n \leq \frac{(24)4^5}{180 \cdot 4^4} = .5333.$$  

(c) How large would $n$ have to be in order to guarantee that $S_n \approx \ln 5$ is accurate to 5 decimal places? You do not need to simplify.

For five decimal places of accuracy, we need $ES_n \leq \frac{1}{10^5}$, i.e.

$$\frac{24 \cdot 4^5}{180n^4} \leq \frac{1}{10^6}, \quad \text{i.e.} \quad 180n^4 \geq 24 \cdot 4^5 \cdot 10^6,$$

i.e. $n \geq \sqrt[4]{\frac{24 \cdot 4^5 \cdot 10^6}{180}} = 108.01$. Because $n$ must be even for Simpson’s rule, we would need to take $n = 110$.

4. (20 pts) A solid is formed by rotating the shaded region around the $y$-axis.

(a) Express the volume of the solid as an integral using $dx$. Do not evaluate the integral.

Draw a thin vertical rectangle of width $dx$ through the region, $x$ units from the $y$-axis. When this is rotated around the $y$-axis, it forms a cylindrical shell:

The volume of the shell is $V_{\text{shell}} = 2\pi rhdx$. In this case, $r = x$, and $h$ is the height of the rectangle: $h = (\text{top coordinate}) - (\text{bottom coordinate}) = \sqrt{25 - x^2} - 4$. The range of $x$ in the region is from 0 to the intersection point with $y = 4$. When $y = 4$, $x = \sqrt{25 - y^2} = \sqrt{25 - 16} = \sqrt{9} = 3$. Thus

$$V = \int_0^3 2\pi x(\sqrt{25 - x^2} - 4)dx.$$
(b) Express the volume of the solid as an integral using $dy$. Do not evaluate the integral.

Using $dy$ results in disks here. The cross-section of the solid at height $y$ is a circle of radius $x = \sqrt{25 - y^2}$. Therefore

$$V = \int_4^5 A(y)dy = \int_4^5 \pi (\sqrt{25 - y^2})^2 dy = \pi \int_4^5 (25 - y^2) dy.$$ 

5. (20 pts) A square well is 10m deep, and each of its four sides has length 2. The well is filled to a depth of 6m with water. Find the work done in pumping all of the water out of the well. (Recall that the density of water is 1000 kg/m$^3$, and that the acceleration due to gravity is 9.8 m/sec$^2$.)

Consider a slice of water at height $y$. This slice has to be pumped a distance of $D = 10 - y$ meters. The force required to lift the slice is

$$F = ma = \text{Vol} \times \text{Density} \times (9.8) = 2 \times 2 \times dy \times 1000 \times 9.8 N.$$ 

The work done pumping the slice is

$$W_{\text{slice}} = F \times D = 4000 \times 9.8(10 - y) dy J.$$ 

We add up these increments of work by integrating. The range of $y$ is given between 0 and 6, so

$$W = \int_0^6 4000 \times 9.8(10 - y) dy = 4000 \times 9.8 \left( 10y - \frac{y^2}{2} \right)_{0}^{6} = 4000 \times 9.8 \left( 60 - \frac{36}{2} \right) J.$$