Part I

1. (12 pts) Compute \( \int \frac{x^2 - 6}{x^2 - 2x} \, dx \).

First use long division to get
\[
\int \frac{x^2 - 6}{x^2 - 2x} \, dx = \int \frac{2x - 6}{x(x - 2)} \, dx.
\]

Now use Partial Fractions:
\[
\frac{2x - 6}{x(x - 2)} = \frac{A}{x} + \frac{B}{x - 2}.
\]

Clearing denominators, we find
\[
2x - 6 = A(x - 2) + Bx.
\]

Set \( x = 0 \) to see that \(-6 = -2A\), so \( A = 3 \). Set \( x = 2 \) to see that \(-2 = 2B\), so \( B = -1 \).

Therefore
\[
\int \frac{x^2 - 6}{x^2 - 2x} \, dx = \int 1 + \frac{3}{x} - \frac{1}{x - 2} \, dx = x + 3 \ln |x| - \ln |x - 2| + C.
\]

2. (15 pts)

(a) Use a Riemann sum with \( n = 4 \) rectangles to approximate \( \int_1^9 (2x + 3)^9 \, dx \).

(You don’t need to simplify).

In this case, \( \Delta x = \frac{9 - 1}{4} = 2 \), so \( x_0 = 1, x_1 = 3, x_2 = 5, x_3 = 7, x_4 = 9 \). We use right endpoints below, but other answers are possible.

\[
R_4 = (6 + 3)^9(2) + (10 + 3)^9(2) + (14 + 3)^9(2) + (18 + 3)^9(2).
\]

(b) Compute the exact value of the integral.

Let \( u = 2x + 3 \), \( du = 2 \, dx \). When \( x = 1 \), \( u = 5 \), and when \( x = 9 \), \( u = 21 \). Thus the integral is equal to
\[
\int_5^{21} \frac{u^9 \, du}{2} = \frac{1}{2} \left[ \frac{u^{10}}{10} \right]_5^{21} = \frac{1}{20} (21^{10} - 5^{10}).
\]
3. (12 pts) Compute $\int x \sin(x) \, dx$

Use IBP with $u = x$ and $dv = \sin(x) \, dx$. Then $du = dx$ and $v = -\cos(x)$, so

$$\int x \sin(x) \, dx = -x \cos(x) - \int (-\cos(x)) \, dx = -x \cos(x) + \sin(x) + C.$$ 

4. (15 pts) The feeding trough pictured below is full of slop. The density of slop is 82 kg/m³. The acceleration due to gravity is 9.8 m/sec².

Find the work done by the pigs in sucking the slop out the top of the trough.

Consider a slice of water at height $y$. By similar triangles, the width of the slice is $2y$. Thus the volume of the slice is

$$V_{\text{slice}} = (2y)(4) \, dy.$$ 

The mass of the slice is the volume times the density. Therefore the force on the slice due to gravity is

$$F_{\text{slice}} = ma = (9.8)(82)(8y) \, dy.$$ 

The work done in lifting this slice to the top is

$$W_{\text{slice}} = F \cdot D = (9.8)(82)(8y)(1-y) \, dy$$

since the distance from the slice to the top is $(1-y)$.

We add up all these incremental pieces of work (one for each $y$) by integrating:

$$W = \int_0^1 (9.8)(82)(8)(y - y^2) \, dy = (9.8)(82)(8)\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = (9.8)(82)(8)\left(\frac{1}{2} - \frac{1}{3}\right) \, \text{Joules}.$$
5. (15 pts) Compute the following integral:

$$\int \frac{x^2}{\sqrt{1-x^2}}^3 dx$$

Write $x = \sin \theta$, so $dx = \cos \theta d\theta$. Then

$$\sqrt{1-x^2} = \sqrt{1-\sin^2 \theta} = \sqrt{\cos^2 \theta} = \cos \theta$$

since we can assume $-\pi/2 \leq \theta \leq \pi/2$. Then

$$\int \frac{x^2}{\sqrt{1-x^2}}^3 dx = \int \frac{\sin^2 \theta}{\cos^3 \theta} \cos \theta d\theta = \int \frac{\sin^2 \theta}{\cos^2 \theta} d\theta = \int \tan^2 \theta d\theta$$

$$= \int (\sec^2 \theta - 1) d\theta = \tan \theta - \theta + C.$$ 

To convert back to $x$, draw a triangle, or note that

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{x}{\sqrt{1-x^2}}$$

Furthermore, $\theta = \arcsin(x)$. Hence

$$\int \frac{x^2}{\sqrt{1-x^2}}^3 dx = \frac{x}{\sqrt{1-x^2}} - \arcsin(x) + C.$$ 

6. (16 pts) Use calculus to prove that a sphere of radius $R$ has volume $V = \frac{4}{3} \pi R^3$. (Recall that the equation of the semicircle is $y = \sqrt{R^2 - x^2}$.)

The volume of a solid oriented along the $x$-axis between $x = a$ and $x = b$ is given by

$$V = \int_a^b A(x) dx,$$

where $A(x)$ is the cross-sectional area of a slice at $x$. In this problem, the cross-sections are circles, so $A(x) = \pi r^2$, where $r$ is the radius. This radius is the height of the circle at $x$, i.e. $r = \sqrt{R^2 - x^2}$ and $A(x) = \pi r^2 = \pi (R^2 - x^2)$. Thus

$$V = \int_{-R}^R \pi (R^2 - x^2) dx = 2\pi \int_0^R (R^2 - x^2) dx = 2\pi \left( R^3 - \frac{x^3}{3} \right) \bigg|_0^R$$

$$= 2\pi (R^3 - (R^3/3)) = \frac{4}{3} \pi R^3.$$
7. (15 pts) Consider the following differential equation: \( \frac{dy}{dx} = e^y \sin x \).

(a) Find the general solution \( y \) to the equation.

We have

\[
e^{-y}dy = \sin x \, dx.
\]

Integrating both sides,

\[
-e^{-y} = -\cos x + C \quad \implies \quad e^{-y} = \cos x - C.
\]

Taking the natural log we find

\[
-y = \ln(\cos x - C),
\]

i.e.

\[
y = -\ln(\cos x - C)
\]

is the general solution.

(b) Find the solution whose graph passes through the point \((0, 0)\).

We set \( x = 0 \) and \( y = 0 \) to find \( C \):

\[
0 = -\ln(1 - C) \quad \implies \quad 0 = \ln(1 - C) \quad \implies \quad 1 = 1 - C \quad \implies \quad C = 0.
\]

Therefore the solution we seek is

\[
y = -\ln(\cos x).
\]

This solution is valid for \( x \in (-\pi/2, \pi/2) \).
Part II

1. (14 pts)

(a) Determine whether the following series is convergent. Show all of your work.

\[
\sum_{n=0}^{\infty} \frac{n^2 + 3}{n^2 - 4n - 5}
\]

This series is divergent because the terms don't go to 0 as \(n \to \infty\) (they go to 1).

(b) Find the sum of the series

\[
\sum_{n=1}^{\infty} a_n = \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \frac{1}{4} + \cdots
\]

(Hint: begin by describing the partial sums.)

Consider the partial sums:

\[ s_1 = \frac{1}{2}, \quad s_2 = 0, \quad s_3 = \frac{1}{3}, \quad s_4 = 0, \quad s_5 = \frac{1}{4}, \quad s_6 = 0, \quad s_7 = \frac{1}{5}, \ldots \]

Thus \( s_n = 0 \) when \( n \) is even, while \( s_n = \frac{1}{n+1} \) when \( n \) is odd. Clearly as \( s_n \to 0 \) as \( n \to \infty \). Hence the series converges to 0. (It is not absolutely convergent though!)

2. (24 pts) Consider the series

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots
\]

(a) Explain why the series is convergent. (Bonus: Find the sum!)

It converges by the alternating series test since \( a_1 > a_2 > a_3 > \cdots \) and \( a_n \to 0 \). In fact, we can recognize the sum as \( \arctan(1) = \pi/4 \).

(b) How many terms of this series would you have to add in order to approximate the sum to within 0.1 = \( \frac{1}{10} \)?

We need \( |a_{n+1}| < \frac{1}{10} \). The first term which is less than 1/10 is the 6th one since \( 1/11 < 1/10 \). Thus we need to add 5 terms.

(c) The above series is: (underline one) Abs Convergent \underline{Cond Convergent}

Justify your answer below by using the integral test.
Let \( f(x) = \frac{1}{2x+1} \). Note \( f(x) \to 0 \) as \( x \to \infty \) and \( f(x) > 0 \). Thus the integral test applies.

\[
\int_1^\infty f(x)\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{2x + 1} \,dx.
\]

Let \( u = 2x + 1 \), so \( du = 2 \,dx \). When \( x = 1 \), \( u = 3 \). When \( x = t \), \( u = 2t + 1 \). Thus the above is

\[
= \lim_{t \to \infty} \frac{1}{2} \int_3^{2t+1} u^{-1} \,du = \lim_{t \to \infty} \frac{1}{2} \ln |u| \bigg|_3^{2t+1} = \lim_{t \to \infty} \frac{1}{2} \ln |2t + 1| - \frac{1}{2} \ln 3 = \infty.
\]

Hence by the integral test, the series \( \sum \frac{1}{2n+1} \) also diverges, and the given series is not absolutely convergent.

3. (22 pts) Let \( f(x) = \frac{1}{1 + 2x^3} \).

(a) Express \( f(x) \) as a power series.

Put \( f(x) \) in the form of the sum of a geometric series \( \frac{a}{1-r} \):

\[
f(x) = \frac{1}{1 - (-2x^3)} = \sum_{n=0}^{\infty} (-2x^3)^n = 1 - 2x^3 + 4x^6 - 8x^9 + \cdots.
\]

(b) For which \( x \) does the above power series converge?

A geometric series converges exactly when \(|r| < 1\). Thus the above equality is valid when

\[
| - 2x^3 | < 1 \implies |x|^3 < \frac{1}{2} \implies |x| < \frac{1}{\sqrt[3]{2}}.
\]

(c) Use your power series to compute \( f^{(6)}(0) \).

The above must coincide with the Maclaurin series of \( f(x) \). This means that the coefficient of \( x^6 \) is

\[
4 = \frac{f^{(6)}(0)}{6!} \implies f^{(6)}(0) = 6! \cdot 4.
\]
4. (20 pts) Consider the power series

\[ \sum_{n=1}^{\infty} \frac{2^n(x - 3)^n}{n + 1}. \]

(a) Find the radius of convergence \( R \) for the power series.

Use the ratio test.

\[
\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}|x - 3|^{n+1}}{n + 2} \cdot \frac{n + 1}{2^n|x - 3|^n} = 2|x - 3| \frac{n + 1}{n + 2}.
\]

Therefore

\[
L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = 2|x - 3| \lim_{n \to \infty} \frac{n + 1}{n + 2} = 2|x - 3|.
\]

By the ratio test, the series converges absolutely when \( L = 2|x - 3| < 1 \), i.e. when \( |x - 3| < 1/2 \). So \( R = 1/2 \).

(b) Find the interval of convergence (i.e. the domain) of the power series.

We know the series converges absolutely on \( |x - 3| < 1/2 \), i.e. for \( x \) in the interval \((3 - 1/2, 3 + 1/2)\). It diverges when \( |x - 3| > 1/2 \). We just have to check the endpoints of the interval. Setting \( x = 3 - 1/2 \), we get the series

\[
\sum_{n=1}^{\infty} \frac{2^n(-\frac{1}{2})^n}{n + 1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n + 1},
\]

which converges by the alternating series test. When \( x = 3 + 1/2 \), we get the series

\[
\sum_{n=1}^{\infty} \frac{2^n(\frac{1}{2})^n}{n + 1} = \sum_{n=1}^{\infty} \frac{1}{n + 1},
\]

which diverges by comparison with the harmonic series. Hence the interval of convergence is \([3 - 1/2, 3 + 1/2)\).
5. (20 pts)

(a) Write down the Maclaurin series for \( f(x) = e^x \).

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\]

(b) Write down the Maclaurin series for \( g(x) = e^{x^2} \).

\[
e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}.
\]

(c) Approximate \( \int_0^1 e^{x^2} \, dx \) by using the first 3 terms of the series in part (b).

\[
\int_0^1 e^{x^2} \, dx \approx \int_0^1 (1 + x^2 + \frac{(x^2)^2}{2!}) \, dx = x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2} \bigg|_0^1 = 1 + (1/3) + (1/10).
\]