1. (20 pts) Set up an integral representing the length of the quarter circle shown below. You do not need to compute the integral.
(Bonus: compute!)

The arclength is

\[ L = \int_0^3 \sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \, dx. \]

Note that

\[ \frac{dy}{dx} = \frac{1}{2} (9 - x^2)^{-1/2} (-2x) = \frac{-x}{\sqrt{9 - x^2}}. \]

Therefore

\[ L = \int_0^3 \sqrt{\frac{x^2}{9 - x^2} + 1} \, dx \quad \text{(full credit for getting this far)} \]

\[ = \int_0^3 \sqrt{\frac{x^2 + 9 - x^2}{9 - x^2}} \, dx = \int_0^3 \sqrt{\frac{9}{9 - x^2}} \, dx = \int_0^3 \frac{3}{\sqrt{9 - x^2}} \, dx \]

This is an improper integral, so it is

\[ = \lim_{t \to 0^+} \int_t^3 \frac{3}{\sqrt{9 - x^2}} \, dx = \lim_{t \to 0^+} 3 \arcsin(x/3) \bigg|_t^3 \]

\[ = \lim_{t \to 0^+} 3 \left( \frac{\pi}{2} - \arcsin(t) \right) = 3\pi/2. \]
2. (20 pts) Consider the region under the graph of \( y = \sqrt{x} \) between \( x = 0 \) and \( x = 4 \):

A solid is formed by rotating this region around the horizontal line \( y = -1 \) (shown dotted).

(a) Using \( dx \) to compute the volume will result in: \textbf{Washers} \hspace{1cm} \textbf{Shells} \hspace{1cm} (circle one).

(b) Using \( dy \) to compute the volume will result in: \textbf{Washers} \hspace{1cm} \textbf{Shells} \hspace{1cm} (circle one).

(c) Compute the volume of the solid, using whichever method you prefer.

Using \( dx \): Cross-sectional area is

\[
A(x) = \pi(R^2 - r^2) = \pi((1 + \sqrt{x})^2 - (1)^2) = \pi(1 + 2\sqrt{x} + x - 1) = \pi(2\sqrt{x} + x).
\]

Therefore the volume is

\[
V = \pi \int_{0}^{4} (2x^{1/2} + x) \, dx = \pi \left( \frac{4}{3} x^{3/2} + \frac{x^2}{2} \right)_{0}^{4} = \pi \left( \frac{4}{3} (4)^{3/2} + \frac{16}{2} \right) = \pi \left( \frac{32}{3} + 8 \right) = \frac{56\pi}{3}.
\]

Using \( dy \): The volume of one shell is

\[
V_{\text{shell}} = 2\pi rh \, dy = 2\pi(y + 1)(4 - y^2) \, dy.
\]

Therefore the volume is

\[
V = 2\pi \int_{0}^{2} (y + 1)(4 - y^2) \, dy = 2\pi \int_{0}^{2} (4y - y^3 + 4 - y^2) \, dy = 2\pi \left( 2y^2 - \frac{y^4}{4} + 4y - \frac{y^3}{3} \right)_{0}^{2}
\]

\[
= 2\pi(8 - 4 + 8 - \frac{8}{3}) = \frac{56\pi}{3}.
\]
3. **(20 pts)** A well is 10m deep, and has a cylindrical shape, with radius 2m. The well contains water up to a height of 5m. Find the work done in pumping all of the water out the top of the well.

A thin slice of water at height \(y\) has to be lifted a distance of \(D = (10 - y)m\). The weight of the slice is \(F = ma = (\text{Vol})(\text{density})(9.8)\). The volume is \(\pi r^2 dy = 4\pi dy\). So the work done in pumping the slice is

\[
W_{\text{slice}} = DF = (10 - y)4\pi(1000)(9.8)dy.
\]

There is one such “slice” for each \(y\) between 0 and 5. The total work is

\[
W = \int_0^5 4\pi(9800)(10 - y)dy = 4\pi(9800) \left(10y - \frac{y^2}{2}\right)_0^5
\]

\[= 4\pi(9800)(50 - \frac{25}{2})\]

where the units are Newtons.

4. **(20 pts)** A block of radioactive kryptonite has radioactive decay. Let \(m(t)\) represent the mass of the block at time \(t\) (in hours). The rate of decay is twice the mass at any given time.

(a) Express the last sentence as a differential equation. (Hint: the rate of decay is \(-\frac{dm}{dt}\).)

\[-\frac{dm}{dt} = 2m.\]

(b) Solve the above differential equation.

\[
\frac{dm}{m} = -2dt \quad \Rightarrow \quad \int \frac{dm}{m} = -2 \int dt
\]

\[\Rightarrow \ln |m| = -2t + C \quad \Rightarrow \quad m = e^{-2t+C}.
\]

Here \(|m| = m\) since \(m > 0\).

(c) Suppose that at time \(t = 0\) the block has a mass of 8 kg. What will the mass be after 2 hours?

Setting \(t = 0\) we see that \(8 = m(0) = e^C\), so \(C = \ln 8\). Therefore

\[m(2) = e^{-4+\ln 8} \text{ kg}.\]
5. (20 pts) The following series are geometric. Determine whether each series converges. If convergent, give the sum. If divergent, say why.

(a) $\sum_{n=1}^{\infty} \frac{11^n}{10^n}$

Divergent geometric series because $r = \frac{11}{10} > 1$.

(b) $\frac{2}{3} + \frac{6}{6} + \frac{18}{12} + \frac{54}{24} + \cdots$

Divergent geometric series since $r = \frac{3}{2} > 1$.

(c) $\sum_{n=5}^{\infty} \left(\frac{-1}{2}\right)^n$

Convergent geometric series with $a = (-1/2)^5$ and $r = (-1/2)$. The sum is

$$S = \frac{a}{1 - r} = \frac{(-1/2)^5}{1 - (-1/2)} = \left(\frac{-1}{2}\right)^5 \cdot \frac{2}{3} = -\frac{1}{3} \cdot 2^4$$