Part I

1. (12 pts) Compute the following integral:

\[ \int \frac{\sqrt{\ln x}}{x} \, dx \]

Let \( u = \ln x \), \( du = \frac{dx}{x} \). The above is

\[ = \int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (\ln x)^{3/2} + C. \]

2. (14 pts) Compute the following integral:

\[ \int \frac{x^2 + x - 1}{x^2 - x} \, dx. \]

This is a rational function. The degree of the numerator is not smaller than that of the denominator, so we first need to divide: \( x^2 + x - 1 \) goes into \( x^2 - x \) one time with a remainder of 2\( x - 1 \). Thus the above is

\[ = \int 1 + \frac{2x - 1}{x^2 - x} \, dx = x + \int \frac{2x - 1}{x^2 - x} \, dx. \]

There are 2 ways to continue. The easiest is to let \( u = x^2 - x \), so \( du = (2x - 1)dx \). This easily gives the answer \( x + \ln |x^2 - x| + C \).

Another (more difficult) solution is to use partial fractions:

\[ \frac{2x - 1}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1} \]

\[ \implies 2x - 1 = A(x - 1) + Bx = (A + B)x - A. \]

Comparing coefficients, we see that \( A = 1 \) and hence \( B = 1 \). We conclude that

\[ \int \frac{x^2 + x - 1}{x^2 - x} \, dx = x + \int \frac{1}{x} \, dx + \int \frac{1}{x - 1} \, dx \]

\[ = x + \ln |x| + \ln |x - 1| + C. \]
3. (16 pts) Suppose you need to evaluate the integral 
\[ \int_0^\pi \sin(x^2) \, dx. \]
This cannot be solved using elementary functions, so it has to be done numerically. Use any of the methods we learned (a Riemann sum, Trapezoidal rule, or Simpson’s rule) to estimate the value of the integral.

Make sure you indicate which method and which \( n \) (your choice!) you are using.

You don’t need to simplify your answer, but the final answer should be a number!

Riemann sum, right endpoints, \( n = 4 \) rectangles, \( \Delta x = \pi/4 \):
\[ R_4 = \sin((\pi/4)^2)(\pi/4) + \sin((\pi/2)^2)(\pi/4) + \sin((3\pi/4)^2)(\pi/4) + \sin(\pi^2)(\pi/4). \]

Trapezoids, \( n = 4 \):
\[ T_4 = \frac{1}{2} \cdot \frac{\pi}{4} \left[ \sin(0) + 2 \sin((\pi/4)^2) + 2 \sin((\pi/2)^2) + 2 \sin((3\pi/4)^2) + \sin(\pi^2) \right]. \]

Simpsons, \( n = 4 \):
\[ S_4 = \frac{1}{3} \cdot \frac{\pi}{4} \left[ \sin(0) + 4 \sin((\pi/4)^2) + 2 \sin((\pi/2)^2) + 4 \sin((3\pi/4)^2) + \sin(\pi^2) \right]. \]

Another way: use the degree \( n = 6 \) Taylor polynomial for \( \sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n(x^2)^{2n+1}}{(2n+1)!} \):
\[ \int_0^\pi \sin(x^2) \, dx \approx \int_0^\pi P_6(x) \, dx = \int_0^\pi (x^2) - \frac{(x^2)^3}{3!} \, dx \]
\[ = \frac{x^3}{3} - \frac{x^7}{3! \cdot 7} \bigg|_0^\pi = \frac{\pi^3}{3} - \frac{\pi^7}{42}. \]

4. (20 pts) Superman is trapped in a cell with 16 grams of radioactive Kryptonite. After 5 hours, there are 4 grams of Kryptonite left.

(a) How much Kryptonite remains after \( t \) hours? Let \( m(t) \) denote the mass (grams) at time \( t \) hours. We know that radioactive decay gives
\[ m(t) = m(0)e^{-kt} = 16e^{-kt}. \]
To find \( k \), set \( t = 5 \): \[ 4 = 16e^{-5k} \implies e^{5k} = 4 \implies 5k = \ln 4 \implies k = \frac{\ln 4}{5}. \] Thus \( m(t) = 16e^{-\ln(4)t/5}. \)

(b) Superman will have enough strength to break out of the cell once only 1 gram of Kryptonite remains. At what time can he escape? Set \( m(t) = 1 = 16e^{-\ln(4)t/5}. \) We find that \( e^{\ln(4)t/5} = 16 \implies \ln(4)t/5 = \ln(16) \implies t = 5\ln(16)/\ln(4) \) hours later.
5. (18 pts) A carpet which is 8 meters long is rolled up. When \( x \) meters have been unrolled, the force required to unroll it further is

\[
F(x) = e^x(8 - x) \text{ Newtons.}
\]

How much work does it take to unroll the entire carpet?

The work is the integral of the force:

\[
W = \int_0^8 e^x(8 - x) \, dx.
\]

Use IBP: \( u = (8 - x), \ du = -dx, \ dv = e^x \, dx, \ v = e^x. \)

\[
W = (8 - x)e^x|_0^8 - \int_0^8 e^x(-dx) = (8 - 8)e^8 - (8 - 0)e^0 + (e^8 - e^0)
\]

\[
= e^8 - 9 \text{ Joules.}
\]

6. (20 pts) Consider the shaded region above the graph of \( y = x^2 \) and below the line \( y = 4 \):

A solid is formed by rotating this region around the \( x \)-axis.

(a) Using \( dx \) to compute the volume will result in: **Washers** Shells (underline one).

(b) Using \( dy \) to compute the volume will result in: **Washers** Shells (underline one).

(c) Compute the volume of the solid, using whichever method you prefer.

Using \( dx \): \[
V = \int_0^2 \pi((4)^2 - (x^2)^2) \, dx = \pi \left(16x - \frac{x^5}{5}\right)|_0^2 = \pi(32 - \frac{32}{5}) = \frac{32}{5} \cdot 4 \pi.
\]

Using \( dy \): \[
V = \int_0^4 2\pi y\sqrt{y} \, dy = 2\pi \int_0^4 y^{3/2} \, dy = 2\pi(2/5)y^{5/2}|_0^4 = 4\pi / 5 \cdot 2^5 = \frac{32}{5} \cdot 4 \pi.
\]
Part II
1. (20 pts) Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. For each series, state which test(s) you are using. Show all of your work.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n} \) Diverges. Harmonic series, \( p \)-series with \( p = 1 \).

(b) \( \sum_{n=1}^{\infty} \frac{n^7 + 4n^4}{12n^{11} + 3n^4 + 1} \) Converges absolutely. Use the limit comparison test with the absolutely convergent \( p \)-series \( \sum b_n = \sum \frac{1}{n^4} \):

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^7 + 4n^4}{12n^{11} + 3n^4 + 1} \cdot \frac{n^4}{1} = \lim_{n \to \infty} \frac{n^{11} + 4n^8}{12n^{11} + 3n^4 + 1} = \lim_{n \to \infty} \frac{n^{11}}{12n^{11}} = \frac{1}{12}.
\]

Since the limit is finite and nonzero, the two series have the same convergence properties.

(c) \( \sum_{n=1}^{\infty} \sin(n) \) This series diverges by the divergence test: the terms do not approach 0.

(d) \( \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \) This series is conditionally convergent. We see that it converges by the alternating series test (the terms are decreasing and approach 0). However if we take the absolute values of the summands, we obtain the \( p \)-series \( \sum \frac{1}{n^{1/2}} \), which diverges since \( p = \frac{1}{2} \leq 1 \).

2. (10 pts) Use the integral test to determine whether the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^3} \]

converges. Use complete sentences.

Let \( f(x) = x^{-3} \). Because \( f \) is positive and decreasing, the given series \( \sum f(n) \) has the same convergence behavior as the improper integral

\[
\int_{1}^{\infty} x^{-3} \, dx = \lim_{t \to \infty} \int_{1}^{t} x^{-3} \, dx = \lim_{t \to \infty} \left[ x^{-2} \right]_{1}^{t} = \lim_{t \to \infty} \frac{1}{-2t^2 + 1} = \frac{1}{2}.
\]

Since the integral converges, the series also converges.
3. (18 pts) Let \( f(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \).

(a) Compute \( T_3(x) \), the Taylor polynomial of degree 3 about \( c = 1 \) for \( f(x) \).

\[
T_3(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3.
\]

We just have to compute the coefficients:

\[
f'(x) = -(1/2)x^{-3/2} \quad f''(x) = (3/2)(1/2)x^{-5/2} \quad f'''(x) = -(5/2)(3/2)(1/2)x^{-7/2},
\]

so plugging \( x = 1 \) into each of the above, we find

\[
T_3(x) = 1 - \frac{1}{2}(x - 1) + \frac{3}{2! \cdot 4}(x - 1)^2 - \frac{15}{3! \cdot 8}(x - 1)^3.
\]

(b) Use \( T_3(x) \) to approximate \( \frac{1}{\sqrt{1.1}} \).

\[
\frac{1}{\sqrt{1.1}} = f(1.1) \approx T_3(1.1) = 1 - \frac{1}{2}(1.1) + \frac{3}{8}(1.1)^2 - \frac{15}{48}(1.1)^3.
\]

4. (18 pts) Consider the power series

\[
\sum_{n=1}^{\infty} \frac{(x - 3)^n}{n5^n}.
\]

(a) Find the radius of convergence \( R \) for the power series.

Use the ratio test:

\[
L = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x - 3|^{n+1}}{(n + 1)5^{n+1}} \cdot \frac{n5^n}{|x - 3|^n} = \lim_{n \to \infty} \frac{n}{n + 1} \frac{|x - 3|}{5} = \frac{|x - 3|}{5}.
\]

By the ratio test, the series converges absolutely when \( L < 1 \) and diverges when \( L > 1 \). Thus we need \( |x - 3| < 5 \), so the radius of convergence is \( R = 5 \).

(b) Find the interval of convergence (i.e. the domain) of the power series.

By the above, we know that the series converges on the interval \( |x - 3| < 5 \), i.e. on the interval \((-2, 8)\). We need to check the endpoints separately. When \( x = -2 \), the given series is

\[
\sum_{n=1}^{\infty} \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.
\]
which converges by the alternating series test. When \( x = 8 \), the series gives

\[
\sum_{n=1}^{\infty} \frac{(5)^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n},
\]

which is the (divergent) harmonic series. Thus the domain of the power series is \([-2, 8)\).

5. (15 pts) Match each function with its MacLaurin series.

A) \( \sum_{n=0}^{\infty} \frac{x^n}{n!} \)

\( \arctan(x) \): E

B) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \)

\( \sin(x) \): B

C) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n} \)

\( \ln(1 + x) \): C

D) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \)

\( e^x \): A

E) \( \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \)

\( \cos(x) \): D

6. (9 pts) Find the sum of the series:

\[
5 - 5 \left( \frac{2}{5} \right) + 5 \left( \frac{2}{5} \right)^2 - 5 \left( \frac{2}{5} \right)^3 + 5 \left( \frac{2}{5} \right)^4 - \cdots
\]

This is a geometric series: \( a + ar + ar^2 + ar^3 + ar^4 + \cdots = \frac{a}{1-r} \) (for \( |r| < 1 \)). Here \( a = 5 \), and \( r = -2/5 \), so the sum is \( \frac{5}{1 - (-2/5)} = \frac{25}{7} \).
7. (10 pts) Give an estimate for the sum of the series

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \]

which is accurate to within \(\frac{1}{1000}\). You do not need to simplify your estimate, but you do need to explain why it has the correct degree of accuracy.

This is an alternating series. We can approximate it with the \(n\)th partial sum, which is within \(|a_{n+1}|\) of the actual sum by the alternating series approximation. What should \(n\) be? We need

\[ |a_{n+1}| = \frac{1}{(n+1)^4} < \frac{1}{1000} \implies (n+1)^4 > 1000. \]

We could solve for \(n\), but calculators are not allowed on exams. So let’s plug in some values of \(n\). When \(n = 4\), \((n+1)^4 = 5^4 = 625\) is not big enough. When \(n = 5\), \((n+1)^4 = 6^4 = (36)^2 = 1296 > 1000\). This shows that we need 5 terms to get the desired accuracy:

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} \approx 1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} \]

is accurate to within \(10^{-3}\).