ON THE DISTRIBUTION OF SATAKE PARAMETERS FOR SIEGEL MODULAR FORMS

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Abstract. We prove a harmonically weighted equidistribution result for the p-th Satake parameters of the family of automorphic cuspidal representations of $\mathrm{PGSp}(2n)$ of fixed weight $\mathtt{k}$ and prime-to-p level $N \to \infty$. The main tool is a new asymptotic Petersson formula for $\mathrm{PGSp}(2n)$ in the level aspect.

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1. Introduction

For a split reductive algebraic group G over a number field F, let $\mathcal{A}(G)$ denote the set of cuspidal automorphic representations of $G(\mathbf{A}_F)$. Each element of $\mathcal{A}(G)$ factorizes as a restricted tensor product $\pi = \otimes_v \pi_v$ of irreducible representations of the local groups $G(F_v)$. If v is a nonarchimedean place of F, then the unramified irreducible representations of $G(F_v)$ are parametrized (via the Satake isomorphism) by the semisimple conjugacy classes in the complex dual group $\widehat{G} = \widehat{G}(\mathbf{C})$. When π_v is unramified, we let

$$t_{\pi_v} \in \widehat{T}/W$$

denote the associated Satake parameter. Here, $\widehat{T} = \widehat{T}(\mathbf{C})$ is a split maximal torus of $\widehat{G}$ and $W = N_G(T)/T$ is the Weyl group of G.

It is of great interest to understand the distribution of the points t_{π_v}, possibly with weights, as π and/or v vary. If π is fixed and v varies, then according to the general Sato-Tate conjecture, the points are expected to be equidistributed relative to some naturally

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defined measure μ_{ST} with support in the maximal compact subgroup of $\widehat{T}$. The support condition reflects the presumed predominance of representations satisfying the Ramanujan conjecture.

Many people have considered the analogous "vertical" question of fixing v and varying π in a family. If G admits discrete series over $\mathbf{R}$, then when the t_{π_v} are unweighted, the relevant measure (for many natural families) is the Plancherel measure at v, [Shin]. Shin and Templier obtained a quantitative version of this result with error bounds. Applications include: (i) a diagonal hybrid where the size of the family and the place v both tend to infinity (the relevant measure being Sato-Tate rather than Plancherel), and (ii) a determination of the distribution of the low-lying zeroes of certain families of automorphic L-functions for G, [ST]. Matz and Templier have recently treated the case of $\mathrm{GL}(n)$, [MT].

When the Satake parameters at a fixed place v are given the harmonic weights that arise naturally in the Petersson/Kuznetsov trace formula, it has been shown for many GL(2) and GL(3) families that they exhibit equidistribution relative to the Sato-Tate measure itself ([Li], [KL3], [BBR], [Z]). In this paper we consider the distribution of harmonically weighted Satake parameters at a fixed place p for the group $G = \mathrm{GSp}(2n)$. For simplicity we work over $\mathbf{Q}$ and assume trivial central character. We consider cuspidal representations of level N with fixed holomorphic discrete series archimedean component of weight $\mathtt{k} > 2n$. We weight each Satake parameter t_{π_p} by the globally defined value

$$w_\pi = \sum_{\varphi \in E_{\mathtt{k}}(\pi)} \frac{|c_\sigma(\varphi)|^2}{\|\varphi\|^2}, \tag{1.1}$$

where $E_{\mathtt{k}}(\pi)$ is a finite orthogonal set of cuspidal Hecke eigenforms giving rise to π, and $c_\sigma(\varphi)$ denotes a Fourier coefficient, defined in (6.2). Weighted in this way, we prove that the parameters become equidistributed relative to a certain measure as $N \to \infty$ (see Theorem 8.3). In contrast to the GL(2) case, the measure depends on p. Subject to a natural hypothesis on the growth of the geometric side of the trace formula (which holds at least when $n = 2$), in Theorem 8.4 we relate the measure to the Sato-Tate measure. In particular, it is supported on the tempered spectrum, and tends to the Sato-Tate measure as $p \to \infty$.

A very similar equidistribution problem has been studied already in the case $n = 2$ by Kowalski, Saha, and Tsimerman ([KST2]), who fix the level $N = 1$ and let the archimedean parameter $\mathtt{k} \to \infty$. Using a formula of Sugano, they were able to form a connection between Satake parameters and Fourier coefficients, the latter of which they control with the intricate Petersson formula for GSp(4) due to Kitaoka [Ki]. They use weights which are certain linear combinations of the ones given in (1.1). Their methods have been adapted to treat the case of higher level $N \to \infty$ by M. Dickson [D]. Very recently, Kim, Wakatsuki and Yamauchi have studied the equidistribution problem for GSp(4) via Arthur's invariant trace formula [KWY]. In all of these works, a quantitative equidistribution statement is proven, with application to the distribution of low-lying zeros of L-functions.

The key technical tools used in [KST2], namely, Kitaoka's formula and Sugano's formula, are not yet available when $n > 2$. Nevertheless, we can apply two simple ideas to treat the higher rank case. The first is to use a Hecke operator as a test function in the relative trace formula to derive a Petersson formula for $\mathrm{GSp}(2n)$ whose spectral side involves two Fourier coefficients (as usual) with the additional inclusion of a Satake parameter. In this way we can access the Satake parameters directly without the use of Sugano's formula. In order to project onto the holomorphic cusp forms of weight $\mathtt{k}$, we use a certain matrix coefficient of the weight $\mathtt{k}$ holomorphic discrete series $\pi_{\mathtt{k}}$ of $\mathrm{GSp}(2n, \mathbf{R})$ as the archimedean component of our test function. This function is computed explicitly in the Appendix (Theorem A.9).

The second idea is to take the limit of the kernel function before integrating. This allows us to avoid computing or estimating all but a few of the orbital integrals that show up on the geometric side. The result is an asymptotic Petersson formula with only a finite sum on the geometric side (Theorem 6.3).† In the simplest case where the Hecke operator is trivial, it is given as follows (see §7.3, where notation is explained in detail).

Theorem 1.1. *Let $\mathcal{B}_{\mathtt{k}}(N)$ be an orthogonal basis of the space of degree n Siegel cusp forms of weight $\mathtt{k} > 2n$ with $2|n\mathtt{k}$, and level group $\Gamma_0(N) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbf{Z}) | \, C \equiv O \mod N\}$. For symmetric positive-definite half-integral matrices σ_1, σ_2,*

$$\lim_{N\to\infty} \frac{1}{\psi(N)} \sum_{F\in\mathcal{B}_{\mathtt{k}}(N)} \frac{a_{\sigma_1}(F)\overline{a_{\sigma_2}(F)}}{\|F\|^2} = \delta_{\mathtt{k}}(\sigma_1,\sigma_2) c_{n\mathtt{k}\sigma_1}, \tag{1.2}$$

where $a_{\sigma_j}(F)$ are Fourier coefficients,

$$c_{n\mathtt{k}\sigma_1} = \frac{(\det \sigma_1)^{\mathtt{k}-(n+1)/2}}{\pi^{n(n-1)/4}(4\pi)^{n(n+1)/2-n\mathtt{k}} \prod_{j=1}^{n} \Gamma(\mathtt{k} - \frac{n+j}{2})},$$

and

$$\delta_{\mathtt{k}}(\sigma_1,\sigma_2) = \sum_{\substack{A\in \mathrm{GL}_n(\mathbf{Z})/\{\pm I_n\} \\ {}^tA\sigma_1 A = \sigma_2}} \det(A)^{\mathtt{k}}. \tag{1.3}$$

Remarks:

(1) When $\mathtt{k}$ is even, $\delta_{\mathtt{k}}(\sigma_1,\sigma_2) = \delta_0(\sigma_1,\sigma_2) = \#\{A \in \mathrm{GL}_n(\mathbf{Z})/\{\pm I_n\} | \, {}^tA\sigma_1 A = \sigma_2\}$. When $\mathtt{k}$ is odd, it may happen that $\delta_{\mathtt{k}}(\sigma_1,\sigma_2) = 0$ even when $\delta_0(\sigma_1,\sigma_2) > 0$ (for example, if σ_1, σ_2 are diagonal). In such cases, the spectral side also vanishes, since $a_{\sigma_j}(F) = 0$ by [Kl, p. 45]. We note that $\delta_{\mathtt{k}}(\sigma_1,\sigma_2)$ does not vanish in general when $\mathtt{k}$ is odd.

(2) When $n = 2$ and $\mathtt{k}$ is even, the above result is shown in [CKM], [D]. (In [CKM, Theorem 1.1, Remark 1.4], it is incorrectly asserted that (1.2) holds when $n = 2$ and $\mathtt{k}$ is odd, but with $\delta_0(\sigma_1,\sigma_2)$ in place of $\delta_{\mathtt{k}}(\sigma_1,\sigma_2)$. This is incompatible with the observations in our first remark.)

(3) $c_{n\mathtt{k}\sigma_1}$ is the constant of proportionality between $a_{\sigma_1}(F)$ and a suitably normalized inner product of F with the σ_1-Poincaré series of weight $\mathtt{k}$ ([M], [Kl, p. 90]).

(4) Although we have highlighted the above special case, the main focus of this paper is on local Satake parameters, which are absent in the above theorem.

Petersson/Kuznetsov trace formulas play a fundamental role in the study of automorphic L-functions. There are well-established methods for GL(2), and to a lesser extent GL(3), but applications to other groups are rare. For a recent example, Blomer has used Kitaoka's formula to compute first and second moments of spinor L-functions of GSp(4), with power-saving error term, [Bl]. It would be of great interest to extend Kitaoka's formula from degree 2 to degree n. The machinery we develop here can form the starting point for such a generalization. Although at present there is no quantification of the error term for finite N if $n > 2$ (the $n = 2$ case is treated in Appendix B), the asymptotic formula is sufficient for obtaining the equidistribution result.

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†After this paper was written, we became aware of [KST1], in which a similar idea is applied to classical Poincaré series.

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2. The Satake transform

Here we recall some basic background about the Satake transform. References include [Gr] and [Sha]. For notation in this section, let G be a split group defined over $\mathbf{Q}_p$, and let T be a (split) maximal torus of G defined over $\mathbf{Q}_p$ and contained in a Borel subgroup $B = TN$ with N unipotent. Let $X^*(T) = \mathrm{Hom}(T, \mathrm{GL}_1)$ denote the lattice of algebraic characters of T, and $X_*(T) = \mathrm{Hom}(\mathrm{GL}_1, T)$ the cocharacter lattice.

For each prime p, let $K_p = G(\mathbf{Z}_p)$, which is a maximal compact subgroup of $G_p = G(\mathbf{Q}_p)$. Let $\mathcal{H}(G_p, K_p)$ be the Hecke algebra of locally constant compactly supported complex-valued bi-K_p-invariant functions on G_p. The Satake transform of an element $f \in \mathcal{H}(G_p, K_p)$ is the function on $T(\mathbf{Z}_p)\backslash T(\mathbf{Q}_p)$ given by

$$\mathcal{S}f(t) = \delta(t)^{1/2} \int_{N(\mathbf{Q}_p)} f(tn)dn. \tag{2.1}$$

Here $\delta(t) = \left|\det \mathrm{Ad}(t)|_{\mathrm{Lie}(N_p)}\right|$ is the modular function for B_p.[‡] By way of motivation for (2.1), suppose

$$\pi_\chi = \mathrm{Ind}_{B_p}^{G_p}(\chi\delta^{1/2})$$

is an unramified representation of G_p, with nonzero K_p-invariant vector ϕ. Then for $f \in \mathcal{H}(G_p, K_p)$, ϕ is an eigenfunction of the operator $\pi_\chi(f)$, with eigenvalue

$$\omega_\chi(f) = \int_{T(\mathbf{Q}_p)} \mathcal{S}f(t)\chi(t)dt. \tag{2.2}$$

The Satake transform is a $\mathbf{C}$-algebra isomorphism

$$\mathcal{S} : \mathcal{H}(G_p, K_p) \longrightarrow \mathcal{H}(T(\mathbf{Q}_p), T(\mathbf{Z}_p))^W, \tag{2.3}$$

where the latter denotes the elements which are fixed by the Weyl group W. Let $\widehat{T} = \mathrm{Hom}(X_*(T), \mathbf{C})$ be the dual group of T. It satisfies

$$X^*(T) \cong X_*(\widehat{T}), \quad X_*(T) \cong X^*(\widehat{T}).$$

Using the algebra isomorphisms

$$\mathcal{H}(T(\mathbf{Q}_p), T(\mathbf{Z}_p))^W \cong \mathbf{C}[X_*(T)]^W \cong \mathbf{C}[X^*(\widehat{T})]^W, \tag{2.4}$$

we may view $\mathcal{S}f$ as a function on any of these three spaces. We explain this in some more detail. The first isomorphism in (2.4) arises by identifying an element of $T(\mathbf{Q}_p)/T(\mathbf{Z}_p)$ with a tuple of integer powers of p, which is of the form $\lambda(p)$ for a unique $\lambda \in X_*(T)$. Thus if we write

$$\mathcal{S}f = \sum_{t \in T(\mathbf{Q}_p)/T(\mathbf{Z}_p)} a_t C_t \in \mathcal{H}(T(\mathbf{Q}_p), T(\mathbf{Z}_p)) \tag{2.5}$$

where $a_t \in \mathbf{C}$ is nonzero for at most finitely many t, and C_t is the characteristic function of the coset t, we can make the identification

$$\mathcal{S}f = \sum_{\lambda \in X_*(T)} a_\lambda \lambda \in \mathbf{C}[X_*(T)],$$

[‡] Later on we will take $G = \mathrm{GSp}(2n)$ and B the Borel subgroup determined by the set of positive roots chosen in §3.2. Then $\delta(t)^{1/2} = p^{-\langle \lambda, \rho \rangle}$ if $t = \lambda(p)$ for $\lambda \in X_*(T)$ and ρ is given by (3.9).

where $a_\lambda = a_{\lambda(p)}$ from (2.5). Fix an isomorphism $X_*(T) \cong X^*(\widehat{T})$ and denote it by $\lambda \mapsto \hat{\lambda}$. Then we may in turn identify $\mathcal{S}f$ with the function

$$\mathcal{S}f = \sum_{\hat{\lambda} \in X^*(\widehat{T})} a_{\hat{\lambda}} \hat{\lambda} \in \mathbf{C}[X^*(\widehat{T})], \tag{2.6}$$

where $a_{\hat{\lambda}} = a_\lambda$.

An unramified character χ of $T(\mathbf{Q}_p)$ as in (2.2) can be identified with the Satake parameter $t_\chi \in \widehat{T}(\mathbf{C})$ determined by

$$\chi(\lambda(p)) = \hat{\lambda}(t_\chi) \qquad \text{for all } \lambda \in X_*(T) \tag{2.7}$$

(cf. [Ca, p. 134, Eq. (3)]). With this notation, using (2.5) and (2.6), and taking $\operatorname{meas}(T(\mathbf{Z}_p)) = 1$, (2.2) becomes

$$\omega_\chi(f) = \sum_\lambda a_\lambda \chi(\lambda(p)) = \sum_{\hat{\lambda}} a_{\hat{\lambda}} \hat{\lambda}(t_\chi) = \mathcal{S}f(t_\chi). \tag{2.8}$$

When $\pi = \pi_\chi$ is given, we write $t_\pi = t_\chi$ for the Satake parameter of π.

Proposition 2.1. *Viewing $\mathcal{S}f$ as a function on $\widehat{T}$ as in* (2.6)*, we have*

$$\overline{\mathcal{S}f} = \mathcal{S}f^*,$$

where $f^*(g) = \overline{f(g^{-1})}$.

Proof. For $t \in \widehat{T}$, we need to show that $\mathcal{S}f(t) = \overline{\mathcal{S}f^*(t)}$. As in (2.7), there exists a unique unramified character χ of $T(\mathbf{Q}_p)$ such that $\chi(\lambda(p)) = \hat{\lambda}(t)$ for all $\lambda \in X_*(T)$. Recalling that $\pi_\chi(f)^* = \pi_\chi(f^*)$, by (2.8) we have, for the spherical unit vector $\phi \in \pi_\chi$,

$$\mathcal{S}f(t) = \omega_\chi(f) = \langle \pi_\chi(f)\phi, \phi\rangle = \langle \phi, \pi_\chi(f^*)\phi\rangle = \overline{\omega_\chi(f^*)} = \overline{\mathcal{S}f^*(t)}.$$ □

3. The symplectic group

3.1. **Definition.** Henceforth, we will denote by G the algebraic group GSp_{2n} defined as follows. For any commutative ring R, let $M_{2n}(R)$ be the set of $2n \times 2n$ matrices with entries in R. Letting O denote the zero-matrix of suitable dimension, and I_n denote the $n \times n$ identity matrix, define

$$J = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix},$$

$$\mathrm{Sp}_{2n}(R) = \{M \in M_{2n}(R) |\ {}^tMJM = J\},$$

$$\mathrm{GSp}_{2n}(R) = \{M \in M_{2n}(R) |\ {}^tMJM = r(M)J \text{ and } r(M) \text{ is a unit in } R\}.$$

Thus a matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to $\mathrm{GSp}_{2n}(R)$ if and only if there exists a similitude $r(M) \in R^*$ such that

$$ {}^tAC = {}^tCA, \quad {}^tBD = {}^tDB, \quad {}^tAD - {}^tCB = r(M)I_n. \tag{3.1}$$

Taking inverses in ${}^tMJM = r(M)J$ shows that ${}^tM = \left(\begin{smallmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{smallmatrix}\right) \in \mathrm{GSp}_{2n}(R)$, and the above relations applied to this matrix give

$$A\,{}^tB = B\,{}^tA, \quad C\,{}^tD = D\,{}^tC, \quad A\,{}^tD - B\,{}^tC = r(M)I_n. \tag{3.2}$$

Note that $\mathrm{Sp}_{2n}(R) = \{M \in \mathrm{GSp}_{2n}(R) |\, r(M) = 1\}$. Define

$$\mathbb{G} = \mathrm{PGSp}_{2n} = \mathrm{GSp}_{2n}/Z,$$

where Z is the center (the set of scalar matrices).

Let $S_n(R)$ be the set of $n\times n$ symmetric matrices over R. The Siegel upper half space is the following set of complex symmetric matrices

$$\mathcal{H}_n = \{X+iY \in S_n(\mathbf{C})|\, X, Y \in S_n(\mathbf{R}), Y>0\},$$

where $Y>0$ means that Y is positive definite. It is a complex vector space of dimension $\frac{n(n+1)}{2}$. Letting

$$\mathrm{GSp}_{2n}(\mathbf{R})^+ = \{M \in \mathrm{GSp}_{2n}(\mathbf{R})|\, r(M)>0\},$$

there is a transitive action of $\mathrm{GSp}_{2n}(\mathbf{R})^+$ on $\mathcal{H}_n$ given by

$$M\cdot \mathfrak{Z} = (A\mathfrak{Z}+B)(C\mathfrak{Z}+D)^{-1} \qquad (\mathfrak{Z}\in\mathcal{H}_n).$$

The stabilizer in $\mathrm{Sp}_{2n}(\mathbf{R})$ of the element $iI_n \in \mathcal{H}_n$ is the compact subgroup

$$K_\infty = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbf{R})|\, A+iB \in \mathrm{U}(n)\right\}, \tag{3.3}$$

where $\mathrm{U}(n)$ is the group of $n\times n$ complex unitary matrices X (so $X^{-1} = {}^t\overline{X}$). Define the Siegel parabolic subgroup

$$\begin{aligned} P(\mathbf{R}) &= \left\{ \begin{pmatrix} A & B \\ O & D \end{pmatrix} \in \mathrm{GSp}_{2n}(\mathbf{R})\right\} \\ &= \left\{ \begin{pmatrix} A & \\ & r\,{}^tA^{-1} \end{pmatrix} \begin{pmatrix} I & S \\ & I \end{pmatrix} |\, A \in \mathrm{GL}_n(\mathbf{R}), r\in\mathbf{R}^*, S\in S_n(\mathbf{R})\right\} \end{aligned} \tag{3.4}$$

(where O denotes the $n\times n$ zero matrix), and set $\mathbb{P} = P/Z$. We recall the decomposition

$$\mathrm{GSp}_{2n}(\mathbf{R}) = P(\mathbf{R})K_\infty.$$

For a prime p, let

$$K_p = \mathrm{PGSp}_{2n}(\mathbf{Z}_p).$$

Haar measure on $\mathrm{PGSp}_{2n}(\mathbf{Q}_p)$ will be normalized so that $\mathrm{meas}(K_p)=1$. For an integer $N>0$, set

$$K_0(N)_p = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in K_p|\, C \equiv O \mod N\mathbf{Z}_p\right\}, \tag{3.5}$$

$$K_0(N) = \prod_{p<\infty} K_0(N)_p.$$

3.2. **Root Data.** We review some standard material to fix notation and terminology that will be used in the sequel. Let F be an algebraically closed field. In this subsection we write G for $G(F)$, and similarly for the other algebraic groups considered. In G, the diagonal subgroup

$$T = \{t = \mathrm{diag}(a_1,\dots,a_n,\tfrac{r}{a_1},\dots,\tfrac{r}{a_n})|\, ra_1\cdots a_n \neq 0\} \tag{3.6}$$

is a split maximal torus. Given $\chi \in X^*(T)$, there exist $k_0,\dots,k_n\in\mathbf{Z}$ such that

$$\chi(t) = r^{k_0}a_1^{k_1}\cdots a_n^{k_n} \tag{3.7}$$

for t as in (3.6). By associating χ with the tuple $(k_0,\dots,k_n)$, we obtain a natural identification $X^*(T)\cong \mathbf{Z}^{n+1}$. Let $e_j\in\mathbf{Z}^{n+1}$ be the $(j+1)$-th standard basis vector $(j=0,\dots,n)$. The set Φ of roots (for the action of T on $\mathrm{Lie}(G)$) consists of

$$\begin{aligned} &\pm(e_j-e_i) \quad (1\le i<j\le n), \\ &\pm(e_0-e_j-e_i) \quad (1\le i\le j\le n). \end{aligned} \tag{3.8}$$

We take the set Φ^+ of positive roots to consist of those which have the + coefficient. The corresponding set of simple roots is

$$\Delta = \{e_{j+1} - e_j | j = 1, \ldots, n-1\} \cup \{e_0 - 2e_n\}.$$

Let

$$\rho = \frac{1}{2} \sum_{\chi \in \Phi^+} \chi = \frac{1}{2} \left(\sum_{1 \le i < j \le n} (e_j - e_i) + \sum_{1 \le i \le j \le n} (e_0 - e_i - e_j) \right).$$

Explicitly,

$$\rho = \frac{n(n+1)}{4} e_0 - n e_1 - (n-1) e_2 - \cdots - e_n. \tag{3.9}$$

The cocharacter lattice is $X_*(T) = \mathrm{Hom}(F^*, T)$. We identify a tuple $\lambda = (\ell_0, \ell_1, \ldots, \ell_n) \in \mathbf{Z}^{n+1}$ with the cocharacter

$$\lambda(a) = \mathrm{diag}(a^{\ell_1}, \ldots, a^{\ell_n}, a^{\ell_0 - \ell_1}, \ldots, a^{\ell_0 - \ell_n}). \tag{3.10}$$

In this way, $X_*(T) \cong \mathbf{Z}^{n+1}$.

The composition of a character with a cocharacter yields a rational homomorphism $F^* \to F^*$, which is necessarily of the form $x \mapsto x^m$ for $m \in \mathbf{Z}$. Thus, we have a natural pairing $X^*(T) \times X_*(T) \to \mathbf{Z}$ given by

$$\chi(\lambda(x)) = x^{\langle \chi, \lambda \rangle}. \tag{3.11}$$

In terms of the coordinates given above, this works out to

$$\langle (k_0, \ldots, k_n), (\ell_0, \ldots, \ell_n) \rangle = k_0 \ell_0 + \cdots + k_n \ell_n. \tag{3.12}$$

Our main group of interest is $\mathbb{G} = \mathrm{PGSp}_{2n}$. Letting $\mathbb{T} = T/Z$ be the maximal torus, we can identify its character lattice $X^*(\mathbb{T})$ with the subset of $X^*(T)$ consisting of all characters which are trivial on Z. Thus,

$$X^*(\mathbb{T}) \cong \{(k_0, k_1, \ldots, k_n) \in \mathbf{Z}^{n+1} | \, 2k_0 + k_1 + \cdots + k_n = 0\}. \tag{3.13}$$

Note that the roots (3.8) belong to this set, so we may identify Φ with the set of roots in $X^*(\mathbb{T})$. The cocharacter lattice $X_*(\mathbb{T})$ can be viewed as the quotient of $X_*(T)$ by the subgroup of cocharacters taking values in Z. In terms of the coordinates above (3.10), we have

$$X_*(\mathbb{T}) \cong \mathbf{Z}^{n+1}/(2, 1, \ldots, 1)\mathbf{Z}. \tag{3.14}$$

The pairing (3.11) makes sense for $(\chi, \lambda) \in X^*(\mathbb{T}) \times X_*(\mathbb{T})$, and the formula (3.12) is independent of the choice of coset representative for λ.

The Weyl group of G, namely

$$W = N_{\mathbb{G}}(\mathbb{T})/Z_{\mathbb{G}}(\mathbb{T}) = N_G(T)/Z_G(T),$$

acts on T (and also $\mathbb{T}$) by conjugation. It is isomorphic to $S_n \ltimes (\mathbf{Z}/2\mathbf{Z})^n$, with the following generators:

$$t \mapsto \mathrm{diag}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}, \tfrac{r}{a_{\sigma(1)}}, \ldots, \tfrac{r}{a_{\sigma(n)}})$$

for σ in the symmetric group S_n, and, for $1 \le i \le n$,

$$t \mapsto \mathrm{diag}(a_1, \ldots, a_{i-1}, \tfrac{r}{a_i}, a_{i+1}, \ldots, a_n, \tfrac{r}{a_1}, \ldots, \tfrac{r}{a_{i-1}}, a_i, \tfrac{r}{a_{i+1}}, \ldots, \tfrac{r}{a_n}).$$

Likewise, W acts faithfully on $X^*(\mathbb{T})$ by

$$w\chi(t) = \chi(wtw^{-1}).$$

The corresponding generators are

$$(k_0, k_1, \ldots, k_n) \mapsto (k_0, k_{\sigma^{-1}(1)}, \ldots, k_{\sigma^{-1}(n)}) \tag{3.15}$$

and

$$(k_0, k_1, \ldots, k_n) \mapsto (k_0 + k_i, , k_1, \ldots, k_{i-1}, -k_i, k_{i+1}, \ldots, k_n). \tag{3.16}$$

Using the pairing (3.11), an action of W on $X_*(\mathbb{T})$ is defined implicitly via

$$\langle w\chi, w\lambda \rangle = \langle \chi, \lambda \rangle .$$

In terms of the ℓ-coordinates in (3.10), the action on $X_*(T)$ of the Weyl element in (3.15) is given by

$$(\ell_0, \ell_1, \ldots, \ell_n) \mapsto (\ell_0, \ell_{\sigma^{-1}(1)}, \ldots, \ell_{\sigma^{-1}(n)}),$$

and the one corresponding to (3.16) is given by

$$(\ell_0, \ell_1, \ldots, \ell_n) \mapsto (\ell_0, \ell_1, \ldots, \ell_{i-1}, \ell_0 - \ell_i, \ell_{i+1}, \ldots, \ell_n).$$

Suppose $\chi, \chi' \in X^*(\mathbb{T})$ correspond respectively to $(k_i), (k_i') \in \mathbf{Z}^{n+1}$ as in (3.13). Then the pairing

$$(\chi, \chi') = \sum_{k=1}^{n} k_i k_i'$$

is W-invariant. (This is easily verified using the relation in (3.13).) For a root $\alpha \in \Phi$, there is a unique coroot $\alpha^\vee \in X_*(\mathbb{T})$ satisfying

$$\langle \chi, \alpha^\vee \rangle = \frac{2(\chi, \alpha)}{(\alpha, \alpha)}$$

for all $\chi \in X^*(\mathbb{T})$. We let $\Phi^\vee \subseteq X_*(\mathbb{T})$ denote the set of coroots. These are given explicitly as follows. Let $f_i \in X_*(\mathbb{T})$ denote the dual of e_i; in (3.14), it is the coset attached to the i-th standard basis vector of $\mathbf{Z}^{n+1}$. Then $(e_j - e_i)^\vee = f_j - f_i$ for $i \neq j$, $(e_0 - e_j - e_i)^\vee = -f_j - f_i$ for $i \neq j$, and $(e_0 - 2e_j)^\vee = -f_j$.

The above gives a description of the root datum $(X^*(\mathbb{T}), \Phi, X_*(\mathbb{T}), \Phi^\vee)$ of $\mathbb{G} = \mathrm{PGSp}_{2n}$. The complex dual group $\widehat{\mathbb{G}}$ (with dual root datum $(X_*(\mathbb{T}), \Phi^\vee, X^*(\mathbb{T}), \Phi)$) is $\mathrm{Spin}(2n+1, \mathbf{C})$. In particular, for the maximal torus $\widehat{\mathbb{T}}$ of $\widehat{\mathbb{G}}$, we have

$$X^*(\widehat{\mathbb{T}}) \cong X_*(\mathbb{T}), \qquad X_*(\widehat{\mathbb{T}}) \cong X^*(\mathbb{T}). \tag{3.17}$$

Now define the positive Weyl chamber

$$\mathcal{C}^+ = \{\lambda \in X_*(\mathbb{T}) |\ \langle \chi, \lambda \rangle \geq 0 \text{ for all } \chi \in \Delta\}. \tag{3.18}$$

We will frequently identify $\mathcal{C}^+$ with its counterpart in $X^*(\widehat{\mathbb{T}}) \cong X_*(\mathbb{T})$.

By definition, an element $\lambda \in X_*(\mathbb{T})$ belongs to $\mathcal{C}^+$ if and only if

$$\langle e_{j+1} - e_j, \lambda \rangle \geq 0 \text{ for } j = 1, \ldots, n-1 \text{ and } \langle e_0 - 2e_n, \lambda \rangle \geq 0.$$

The above holds if and only if, in the notation of (3.14), every coset representative $(\ell_0, \ell_1, \ldots, \ell_n)$ for λ satisfies

$$\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n \leq \ell_0/2. \tag{3.19}$$

In fact, each such λ has a unique representative satisfying

$$0 = \ell_1 \leq \ell_2 \leq \cdots \leq \ell_n \leq \ell_0/2. \tag{3.20}$$

For notational convenience, we will often identify λ with this coset representative.

Proposition 3.1 (Cartan decomposition, [Gr]). *The group $\mathbb{G}(\mathbf{Q}_p)$ is the disjoint union of the double cosets $K_p\lambda(p)K_p$ for $\lambda \in \mathcal{C}^+$. The analogous statement (involving λ satisfying (3.19) rather than (3.20)) also holds for $G(\mathbf{Q}_p)$.*

Proposition 3.2. *For $g \in G(\mathbf{Q}_p)$ and $m \geq 1$, let $d_m(g)$ denote the generator (chosen as a power of p) of the fractional ideal of $\mathbf{Q}_p$ generated by the set*

$$\{\det B \,|\, B \text{ is an } m \times m \text{ submatrix of } g\}.$$

Then given $\lambda \in X_*(T)$ satisfying (3.19), an element $g \in G(\mathbf{Q}_p)$ belongs to the double coset $G(\mathbf{Z}_p)\lambda(p)G(\mathbf{Z}_p)$ if and only if each of the following holds:

1. *$r(g) = p^{\ell_0}$*
2. *for each $m = 1, \ldots, n$, $d_m(g) = p^{\ell_1+\cdots+\ell_m}$.*

Proof. This result can be extracted from Chapter II of Newman [Ne]. For convenience, we sketch some of the details. Let $g \in G(\mathbf{Q}_p)$. By the Cartan decomposition, there exist $k_1, k_2 \in G(\mathbf{Z}_p)$ such that $k_1 g k_2 = \lambda(p)$ for a unique cocharacter λ satisfying (3.19). We need to show that the two conditions given above are satisfied. The converse will then also follow, since $\lambda(p)$ is uniquely determined by its first n diagonal entries and its similitude.

The first condition is immediate. For the second, observe that there exists an integer $a \geq 0$ such that $p^a g \in M_{2n}(\mathbf{Z}_p)$. The m-th diagonal coordinate of $k_1 p^a g k_2$ is then $p^{a+\ell_m} \in \mathbf{Z}_p$. Regarding $p^a g$ as an element of $M_{2n}(\mathbf{Z}_p)$ and regarding k_1, k_2 as elements of $\mathrm{GL}_{2n}(\mathbf{Z}_p)$, by [Ne, Chap. II, Sect. 16, Eq. (13)] with $R = \mathbf{Z}_p$, we have

$$p^{a+\ell_m} = \frac{d_m(p^a g)}{d_{m-1}(p^a g)} = \frac{p^{am} d_m(g)}{p^{a(m-1)} d_{m-1}(g)} = p^a \frac{d_m(g)}{d_{m-1}(g)}$$

(with $d_0(g) = 1$). Hence

$$p^{\ell_m} = \frac{d_m(g)}{d_{m-1}(g)}.$$

This is easily seen to be equivalent to condition (2), as needed. □

4. Adelic Siegel modular forms

Let $\mathbf{A}$ denote the adele ring of $\mathbf{Q}$, and fix a Haar measure dg on $\mathbb{G}(\mathbf{A})$. Let $L^2 = L^2(\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A}))$ be the space of measurable functions $\phi : G(\mathbf{A}) \to \mathbf{C}$ satisfying

- $\phi(z\gamma g) = \phi(g)$ for all $z \in Z(\mathbf{A})$, $\gamma \in G(\mathbf{Q})$, $g \in G(\mathbf{A})$
- $\displaystyle\int_{\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})} |\varphi(g)|^2 dg < \infty.$

For any parabolic subgroup P of G, P can be written as MN, where M is the Levi subgroup and N is unipotent. An element $\varphi \in L^2$ is cuspidal if for any parabolic subgroup $P = MN$ of G,

$$\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \varphi(ng)dn = 0 \quad \text{for a.e. } g \in G(\mathbf{A}).$$

We let $L^2_0 \subseteq L^2$ denote the subspace of cuspidal functions.

The right regular representation of $G(\mathbf{A})$ on L^2_0 decomposes discretely as $\bigoplus \pi$, where π are (by definition) the cuspidal automorphic representations of $G(\mathbf{A})$. Any such constituent π is a restricted tensor product

$$\pi = \bigotimes_{p \leq \infty} \pi_p$$

where π_p is an irreducible admissible representation of $G(\mathbf{Q}_p)$.

Fix an integer $\mathtt{k} > n$ with $n\mathtt{k}$ even (so that (A.24) is trivial). Then as in §A.7, there is a holomorphic discrete series representation $\pi_{\mathtt{k}}$ of $\mathbb{G}(\mathbf{R})$ of weight $\mathtt{k}$. Up to unitary scaling, $\pi_{\mathtt{k}}$ contains a unique holomorphic unit vector ϕ_0 satisfying

$$\pi_{\mathtt{k}}(\begin{pmatrix} A & B \\ -B & A \end{pmatrix})\phi_0 = \det(A+Bi)^{\mathtt{k}}\phi_0 \qquad (\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_\infty). \tag{4.1}$$

Let $\Pi_{\mathtt{k}}(N)$ denote the set of cuspidal representations π of $G(\mathbf{A})$ for which $\pi_\infty = \pi_{\mathtt{k}}$ and $\pi_{\mathrm{fin}}^{K_0(N)} \neq 0$. For such π, we let $v_{\pi_\infty} \in V_{\pi_\infty}$ denote a lowest weight vector as in (4.1). For any representation π_{fin} of $G(\mathbf{A}_{\mathrm{fin}})$ and any subgroup $U < G(\mathbf{A}_{\mathrm{fin}})$, we write π_{fin}^U for the space of U-fixed vectors in the space of π_{fin}. Define

$$A_{\mathtt{k}}(N) = \bigoplus_{\pi \in \Pi_{\mathtt{k}}(N)} \mathbf{C}v_{\pi_\infty} \otimes \pi_{\mathrm{fin}}^{K_0(N)}. \tag{4.2}$$

This corresponds to a classical space of holomorphic Siegel cusp forms of weight $\mathtt{k}$ and level N (see §7.3 below).

For $\pi \in \Pi_{\mathtt{k}}(N)$, let $E_{\mathtt{k}}(\pi, N)$ be an orthogonal basis for the summand indexed by π in (4.2). Then the set

$$E_{\mathtt{k}}(N) = \bigcup_{\pi \in \Pi_{\mathtt{k}}(N)} E_{\mathtt{k}}(\pi, N) \tag{4.3}$$

is an orthogonal basis for $A_{\mathtt{k}}(N)$.

5. The test function and its kernel

Any function $f \in L^1(\mathbb{G}(\mathbf{A}))$ defines an operator $R(f)$ on L^2 by

$$R(f)\phi(x) = \int_{\mathbb{G}(\mathbf{A})} f(g)\phi(xg)dg.$$

In this section we will define the bi-$K_0(N)$-invariant test function f to be used in the trace formula.

5.1. **Definition of the test function.** We define, for $g \in G(\mathbf{R})$,

$$f_\infty(g) = d_{\mathtt{k}}\overline{\langle \pi_{\mathtt{k}}(g)\phi_0, \phi_0\rangle},$$

where ϕ_0 is a unit vector in the space of $\pi_{\mathtt{k}}$ satisfying (4.1), and $d_{\mathtt{k}}$ is the formal degree of $\pi_{\mathtt{k}}$. The matrix coefficient is independent of the choice of Haar measure on $\mathbb{G}(\mathbf{R})$. It is computed explicitly in Corollary A.10 of the Appendix. The formal degree $d_{\mathtt{k}}$, which depends on the choice of Haar measure, is given in (A.21). For our purposes, the particular choice of measure is immaterial.

By Proposition A.6, $f_\infty \in L^1(\mathbb{G}(\mathbf{R}))$ precisely when

$$\mathtt{k} > 2n,$$

so this hypothesis will be in force throughout.

We will take $f_{\mathrm{fin}} = \prod_{p<\infty} f_p$ to be a bi-$Z(\mathbf{A}_{\mathrm{fin}})K_0(N)$-invariant function on $G(\mathbf{A}_{\mathrm{fin}})$, of the following form.[†] Fix a finite set $\mathbf{S}$ of prime numbers not dividing N. For $p \in \mathbf{S}$, let f_p be a bi-Z_pK_p-invariant function with compact support modulo Z_p, taking the value 1 on its support. Here, $Z_p = Z(\mathbf{Q}_p)$ is the center of $G(\mathbf{Q}_p)$. By the Cartan decomposition, f_p is the

[†]Although we have defined K_p and $K_0(N)_p$ in (3.5) as subsets of $\mathbb{G}(\mathbf{Q}_p)$, in this section we will blur the distinction between these sets and their preimages in $G(\mathbf{Z}_p)$. No confusion should occur since everything is invariant under the center.

characteristic function of a set of the form Z_pC_p, with C_p a finite union of double cosets of the form $K_p\lambda(p)K_p$, where, by (3.20),

$$\lambda(p) = \mathrm{diag}(p^{\ell_1}, \dots, p^{\ell_n}, p^{\ell_0-\ell_1}, \dots, p^{\ell_0-\ell_n})$$

with $0 = \ell_1 \leq \cdots \leq \ell_n \leq \ell_0/2$. Without any real loss of generality, we make the further assumption that the similitude of C_p has constant valuation, i.e.

$$r(C_p) = p^{r_p}\mathbf{Z}_p^*$$

for some integer $r_p \geq 0$. This amounts to requiring that $\ell_0 = r_p$ for each of the $\lambda(p)$ out of which C_p is built. In particular, $C_p = K_p$ if $r_p = 0$. Having fixed such f_p for each $p \in \mathbf{S}$, we define the global similitude

$$r = \prod_{p\in\mathbf{S}} p^{r_p} \geq 1. \tag{5.1}$$

Now for $p \notin \mathbf{S}$, define $C_p = K_0(N)_p$, and set

$$\psi(N)_p = \mathrm{meas}(K_0(N)_p)^{-1} = [K_p : K_0(N)_p],$$

$$\psi(N) = \prod_p \psi(N)_p = [K_{\mathrm{fin}} : K_0(N)].$$

We then define $f_p : G(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ by

$$f_p(g) = \begin{cases} \psi(N)_p & \text{if } g \in Z_pC_p \\ 0 & \text{otherwise.} \end{cases}$$

(In fact this holds as well when $p \in \mathbf{S}$, since $\psi(N)_p = 1$ in that case.)

Having fixed f_{fin} above, we set

$$f = f_\infty \times f_{\mathrm{fin}}.$$

By our assumption that $\mathtt{k} > 2n$, $f \in L^1(\mathbb{G}(\mathbf{A}))$. The following is a useful observation about the support of f_{fin}.

Proposition 5.1. *Suppose $x, y \in G(\mathbf{A}_{\mathrm{fin}})$ satisfy $r(x)^{-1}r(y) \in \widehat{\mathbf{Z}}^*$. Then for $\gamma \in G(\mathbf{Q})$, $f_{\mathrm{fin}}(x^{-1}\gamma y) \neq 0$ only if there exists $s \in \mathbf{Q}^*$, uniquely determined up to its sign, such that*

$$r(\gamma) = \pm s^2 r. \tag{5.2}$$

Suppose $\gamma \in G(\mathbf{Q})$ satisfies (5.2), and set $\tilde{\gamma} = s^{-1}\gamma$. Then $r(\tilde{\gamma}) = \pm r$, and $f_{\mathrm{fin}}(x^{-1}\gamma y) \neq 0$ if and only if

$$x^{-1}\tilde{\gamma}y \in \prod_p C_p = \prod_{p\in\mathbf{S}} C_p \prod_{p|N} K_0(N)_p \prod_{p\nmid N, p\notin\mathbf{S}} K_p \subseteq M_{2n}(\widehat{\mathbf{Z}}). \tag{5.3}$$

Proof. If $f_{\mathrm{fin}}(x^{-1}\gamma y) \neq 0$, then $x^{-1}\gamma y = zc$ for some $z \in Z(\mathbf{A}_{\mathrm{fin}})$ and $c \in \prod_p C_p$. Since $\mathbf{A}_{\mathrm{fin}}^* = \mathbf{Q}^*\widehat{\mathbf{Z}}^*$, we may write $z = sa$ where $s \in \mathbf{Q}^*$ and $a \in \widehat{\mathbf{Z}}^*$. We may absorb a into c so that $z = s$ without loss of generality. Taking the similitude and using $r(x)^{-1}r(y) \in \widehat{\mathbf{Z}}^*$, we see that $r(\gamma) = s^2ur$ for some $u \in \widehat{\mathbf{Z}}^*$. Since $r(\gamma), s, r \in \mathbf{Q}^*$, it follows that $u \in \widehat{\mathbf{Z}}^* \cap \mathbf{Q}^* = \{\pm 1\}$, as claimed. It is clear that s is unique up to its sign, and that $x^{-1}\tilde{\gamma}y \in \prod_p C_p$. Conversely, since the support of f_{fin} is $Z(\mathbf{A}_{\mathrm{fin}})\prod_p C_p$, it is also clear that if $x^{-1}\tilde{\gamma}y \in \prod_p C_p$, then $f_{\mathrm{fin}}(x^{-1}\gamma y) \neq 0$. $\square$

5.2. **Spectral properties of $R(f)$.** Here we show that the Hecke operator $R(f)$ has finite rank, and we compute its effect on adelic Siegel cusp forms.

Proposition 5.2. *Let U be a unipotent subgroup of $\mathbb{G}$ and let $g \in \mathbb{G}(\mathbf{R})$. Then for almost all $x \in U(\mathbf{R})\backslash\mathbb{G}(\mathbf{R})$,*

$$\int_{U(\mathbf{R})} f_\infty(gux)du = 0.$$

Proof. Let U' be a one-dimensional subgroup of U. There exists $E \in M_{2n}(\mathbf{R})$ such that $U'(\mathbf{R}) = \{I + tE \,|\, t \in \mathbf{R}\}$. By Corollary A.10, there exist complex numbers A, B and $C \neq 0$ depending on $u \in U(\mathbf{R})$, $g, x \in \mathbb{G}(\mathbf{R})$, and $\mathtt{k} > 2n$, such that

$$f_\infty(g(I_n + tE)ux) = \frac{C}{(At+B)^{\mathtt{k}}}.$$

The denominator is nonzero for $t \in \mathbf{R}$ since the matrix coefficient is finite. If $A \neq 0$, then by the fundamental theorem of calculus,

$$\int_{\mathbf{R}} f_\infty(g(I_n + tE)ux)dt = \tfrac{C}{A}(At+B)^{-\mathtt{k}+1}\big|_{-\infty}^{\infty} = 0.$$

If $A = 0$, then

$$\int_{\mathbf{R}} f_\infty(g(I_n + tE)ux)dt = \infty.$$

Hence

$$\int_{U(\mathbf{R})} f_\infty(gux)du = \int_{U'(\mathbf{R})\backslash U(\mathbf{R})} \int_{U'(\mathbf{R})} f_\infty(gu'ux)du'du$$

is either 0 or divergent. It remains to show that this integral is convergent for almost all $x \in U(\mathbf{R})\backslash\mathbb{G}(\mathbf{R})$. But this is immediate from the fact that because $f_\infty \in L^1(\mathbb{G}(\mathbf{R}))$, the integral

$$\int_{\mathbb{G}(\mathbf{R})} f_\infty(gx)dx = \int_{U(\mathbf{R})\backslash\mathbb{G}(\mathbf{R})} \int_{U(\mathbf{R})} f_\infty(gux)du\,dx$$

is convergent. □

Proposition 5.3. *For the test function f defined in §5.1, and the subspace $A_{\mathtt{k}}(N) \subseteq L^2_0$ given in* (4.2)*, $R(f)$ annihilates $A_{\mathtt{k}}(N)^\perp$ and maps $A_{\mathtt{k}}(N)$ into itself.*

Proof. Complete details for the case of GL(2) are given in [KL1, Propositions 13.11, 13.12], and everything carries over directly to the case under consideration here, using the above proposition in place of [KL1, Corollary 13.10]. So we just briefly sketch the ideas. It follows easily from the above proposition that for any $\phi \in L^2$, $R(f)\phi$ is cuspidal, i.e. $R(f) : L^2 \to L^2_0$. Furthermore one checks that f_∞ is self-dual, so the adjoint $R(f)^*$ also has this property. It then follows that $R(f)$ annihilates $(L^2_0)^\perp$. Next, one may use a general orthogonality property of discrete series matrix coefficients, together with the $K_0(N)$-invariance of f_{fin}, to show that the image of $R(f)$ (and of $R(f)^*$) lies in $A_{\mathtt{k}}(N)$. □

It remains to compute the effect of $R(f)$ on a nonzero element $v \in A_{\mathtt{k}}(N)$. We may assume without loss of generality that v is a pure tensor in some cuspidal representation $\pi \in \Pi_{\mathtt{k}}(N)$. Write $\pi = \pi_{\mathtt{k}} \otimes \pi' \otimes \bigotimes_{p\in\mathbf{S}} \pi_p$, where π' is a representation of $\prod'_{p\notin\mathbf{S}} G(\mathbf{Q}_p)$. Accordingly, we write $f = f_\infty \times f' \times \prod_{p\in\mathbf{S}} f_p$ and

$$v = v_\infty \otimes v' \otimes \bigotimes_{p\in\mathbf{S}} v_p,$$

where $v_\infty = \phi_0$ as in (4.1). Then (e.g. by [KL1, Prop. 13.17]) we have

$$R(f)v = \pi_{\mathtt{k}}(f_\infty)v_\infty \otimes \pi'(f')v' \otimes \bigotimes_{p\in\mathbf{S}} \pi_p(f_p)v_p.$$

By the orthogonality relations for discrete series matrix coefficients ([KL1, Corollary 10.26]),

$$\pi_{\mathtt{k}}(f_\infty)v_\infty = v_\infty,$$

where $\pi_{\mathtt{k}}(f_\infty)$ is defined using the same Haar measure as that defining $d_{\mathtt{k}}$. Likewise, because f' is the characteristic function of $K_0(N)' = \prod_{p\notin\mathbf{S}} K_0(N)_p$ scaled by $\mathrm{meas}(K_0(N)')^{-1}$, and $v' \in \pi'^{K_0(N)'}$, we have

$$\pi'(f')v' = v'.$$

For $p \in \mathbf{S}$, since f_p is K_p bi-invariant, $\pi_p(f_p)$ preserves the subspace $\pi_p^{K_p} = \mathbf{C}v_p$. Hence v_p is an eigenvector. Writing $(\mathcal{S}f_p)(t_{\pi_p})$ for the eigenvalue as in (2.8), we have

$$R(f)v = \left(\prod_{p\in\mathbf{S}} (\mathcal{S}f_p)(t_{\pi_p})\right) v \tag{5.4}$$

for v as above.

5.3. **The kernel function.** For the test function f defined in §5.1, the associated kernel function on $G(\mathbf{A}) \times G(\mathbf{A})$ is defined as

$$K(x,y) = K_f(x,y) = \sum_{\gamma\in\mathbb{G}(\mathbf{Q})} f(x^{-1}\gamma y).$$

It satisfies

$$R(f)\phi(x) = \int_{\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})} K(x,y)\phi(y)dy \qquad (\phi \in L^2).$$

Proposition 5.4. *$K(x,y)$ is continuous in both variables. Furthermore, given any subsets $J_1, J_2 \subseteq G(\mathbf{A})$ each having compact image in $\mathbb{G}(\mathbf{A})$, for every $\gamma \in \mathbb{G}(\mathbf{Q})$ there exists a real number α_γ, independent of N and $\mathtt{k}$, such that for all $g_1 \in J_1$ and $g_2 \in J_2$,*

$$d_{\mathtt{k}}^{-1}\psi(N)^{-1}|f(g_1^{-1}\gamma g_2)| \le \alpha_\gamma$$

and

$$\sum_{\gamma\in\mathbb{G}(\mathbf{Q})} \alpha_\gamma < \infty.$$

Proof. For $g \in \mathbb{G}(\mathbf{R})$, define $f_{\mathtt{k}}(g) = \overline{\langle \pi_{\mathtt{k}}(g)\phi_0, \phi_0\rangle}$, so that $f_\infty(g) = d_{\mathtt{k}} f_{\mathtt{k}}(g)$. Define the function $\tilde{f} = \tilde{f}_\infty \tilde{f}_{\mathrm{fin}}$ on $G(\mathbf{A})$, where $\tilde{f}_\infty = d_{\mathtt{k}_0} f_{\mathtt{k}_0}$ with

$$\mathtt{k}_0 = 2n + 1 \le \mathtt{k},$$

and $\tilde{f}_{\mathrm{fin}}$ is the characteristic function of $\prod_p Z_p C_p$, where $C_p = K_p$ for all $p \notin \mathbf{S}$. In other words, $\tilde{f}$ is the test function we have defined earlier in the special case where $\mathtt{k} = \mathtt{k}_0$ and $N = 1$. We claim that

$$\psi(N)^{-1} d_{\mathtt{k}}^{-1} |f(g)| \le d_{\mathtt{k}_0}^{-1} |\tilde{f}(g)|. \tag{5.5}$$

Since $\mathrm{supp}(f_{\mathrm{fin}}) \subseteq \mathrm{supp}(\tilde{f}_{\mathrm{fin}})$, it is clear from the definitions that $\psi(N)^{-1}|f_{\mathrm{fin}}(g)| \le |\tilde{f}_{\mathrm{fin}}(g)|$ for all $g \in G(\mathbf{A}_{\mathrm{fin}})$. For the archimedean part, by Corollary A.10, when $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{G}(\mathbf{R})$ we have

$$\left[\frac{r(g)^{n/2}2^n}{|\det(A + D + i(B - C))|}\right]^{\mathtt{k}} = |f_{\mathtt{k}}(g)| = |\langle \pi_{\mathtt{k}}(g)\phi_0, \phi_0\rangle| \le \|\pi_{\mathtt{k}}(g)\phi_0\|\|\phi_0\| = 1,$$

where we have used the fact that ϕ_0 is a unit vector and $\pi_{\mathtt{k}}$ is unitary. It follows that the expression in the brackets is at most 1, and hence $|f_{\mathtt{k}}(g)| \leq |f_{\mathtt{k}_0}(g)|$ for all $g \in \mathbb{G}(\mathbf{R})$. This proves (5.5).

Hence, it suffices to prove the assertion for $f = \tilde{f}$ (with $N = 1$). By [Li, Prop. 3.1], it suffices to show that there exist bounded compactly supported functions ψ_1, ψ_2 on $\mathbb{G}(\mathbf{A})$ such that $\tilde{f} = \tilde{f} * \psi_1 = \psi_2 * \tilde{f}$, where convolution is defined by

$$f_1 * f_2 = \int_{\mathbb{G}(\mathbf{A})} f_1(\alpha) f_2(\alpha^{-1} g) d\alpha = \int_{\mathbb{G}(\mathbf{A})} f_1(g\alpha^{-1}) f_2(\alpha) d\alpha.$$

For the finite part of ψ_j, we take the characteristic function of $Z(\mathbf{A}_{\text{fin}})K_0(N)$, multiplied by the reciprocal of the measure of this set in $\mathbb{G}(\mathbf{A}_{\text{fin}})$. It remains to define the archimedean components of ψ_1, ψ_2. We claim first that there exists a measurable compactly supported function ξ on $\mathbb{G}(\mathbf{R})$ satisfying

$$\xi(\begin{pmatrix} U & V \\ -V & U \end{pmatrix} g) = \det(U + iV)^{\mathtt{k}_0} \xi(g)$$

for all $\left(\begin{smallmatrix} U & V \\ -V & U \end{smallmatrix}\right) \in K_\infty$, or equivalently, since $G(\mathbf{R}) = K_\infty P(\mathbf{R})$,

$$\xi(\begin{pmatrix} U & V \\ -V & U \end{pmatrix} \begin{pmatrix} A & B \\ O & D \end{pmatrix}) = \det(U + iV)^{\mathtt{k}_0} \xi(\begin{pmatrix} A & B \\ O & D \end{pmatrix}).$$

Let $P' \subseteq P(\mathbf{R})$ be a set of representatives for $(K_\infty \cap P(\mathbf{R}))\backslash P(\mathbf{R})$. Then every element $g \in G(\mathbf{R})$ has a unique decomposition $g = kb'$ with $k \in K_\infty$ and $b' \in P'$. We may choose P' so that it is a real manifold, and then take ξ to be any compactly supported function on P', extended to $G(\mathbf{R})$ by $\xi(kb') = \det(k)^{\mathtt{k}_0} \xi(b')$. One choice of P' is given as follows. Notice that

$$K_\infty \cap P(\mathbf{R}) = \{\begin{pmatrix} U & \\ & U \end{pmatrix} | \, U\,{}^tU = I\} \cong \mathrm{O}(n),$$

so using the decomposition (3.4) of $P(\mathbf{R})$, we see that

$$(K_\infty \cap P(\mathbf{R}))\backslash P(\mathbf{R}) \cong \Big(\mathrm{O}(n) \backslash \mathrm{GL}_n(\mathbf{R})\Big) \times \mathbf{R}^* \times S_n(\mathbf{R}).$$

Hence we can take $P' \subseteq P(\mathbf{R})$ to be the subgroup identifying as

$$P' \cong \left\{ \begin{pmatrix} a_{11} & * & \cdots & * \\ & a_{22} & \cdots & * \\ & & \ddots & * \\ & & & a_{nn} \end{pmatrix} | \, a_{jj} > 0 \right\} \times \mathbf{R}^* \times S_n(\mathbf{R}).$$

The proof now proceeds exactly as in [Li, Prop. 3.2]: one sees easily that $\pi_{\mathtt{k}_0}(\xi)\phi_0$ is a vector of weight $\mathtt{k}_0$ as in (4.1), and hence must be a multiple of ϕ_0. For an appropriate choice of ξ, the multiple is nonzero, and hence without loss of generality $\pi_{\mathtt{k}_0}(\xi)\phi_0 = \phi_0$. From here it is easy to show directly that $\xi * f_{\mathtt{k}_0} = f_{\mathtt{k}_0}$, so we can take $\psi_1 = \xi \times \psi_{\text{fin}}$. Similarly, we can take $\psi_2 = \psi_1^*$. □

5.4. **Spectral expression for the kernel.** Because the support of the test function is not compact modulo the center, some care is needed in order to justify the spectral expansion of the kernel function.

Proposition 5.5. *With notation as in §4, the kernel function of the operator $R(f)$ has the spectral expansion*

$$K(x, y) = \sum_{\pi \in \Pi_{\mathtt{k}}(N)} \Big(\sum_{\varphi \in E_{\mathtt{k}}(\pi, N)} \frac{\varphi(x)\overline{\varphi(y)}}{\|\varphi\|^2} \Big) \prod_{p \in \mathbf{S}} (\mathcal{S} f_p)(t_{\pi_p}). \tag{5.6}$$

Proof. As shown in §5.2, the operator $R(f)$ vanishes on $A_{\mathtt{k}}(N)^{\perp}$ and is diagonalizable on $A_{\mathtt{k}}(N)$, the elements of $E_{\mathtt{k}}(N)$ (see (4.3)) being eigenvectors. It follows easily that

$$\Phi(x,y) = \sum_{\varphi \in E_{\mathtt{k}}(N)} \frac{R(f)\varphi(x)\overline{\varphi(y)}}{\|\varphi\|^2}$$

is a kernel function for $R(f)$. (See e.g. [KL1, p. 228] for details.) It follows that $K(x,y) = \Phi(x,y)$ a.e. On the other hand, Φ is continuous in both variables, being a sum of finitely many adelic Siegel cusp forms, while by Proposition 5.4, $K(x,y)$ is also continuous. Hence they are equal everywhere. Using (5.4) we see that $\Phi(x,y)$ is equal to the spectral expression given in (5.6). □

6. Asymptotic Fourier trace formula for GSp($2n$)

6.1. **Additive characters.** Let

$$\theta : \mathbf{Q}\backslash\mathbf{A} \longrightarrow \mathbf{C}^*$$

be the nontrivial character whose local components are given by

$$\theta_p(x) = \begin{cases} e^{2\pi i x} & \text{if } p = \infty \\ e^{-2\pi i r_p(x)} & \text{if } p < \infty, \end{cases} \tag{6.1}$$

where $r_p(x) \in \mathbf{Q}$ is any number with p-power denominator satisfying $x \in r_p(x) + \mathbf{Z}_p$. All characters of the additive group $\mathbf{Q}\backslash\mathbf{A}$ are of the form $\theta(qx)$ for some $q \in \mathbf{Q}$. It follows easily that any character of the additive group $S_n(\mathbf{Q})\backslash S_n(\mathbf{A})$ has the form $S \mapsto \theta(\operatorname{tr}\sigma S)$ for some $\sigma \in S_n(\mathbf{Q})$. We fix two such matrices $\sigma_1, \sigma_2 \in S_n(\mathbf{Q})$ and define

$$\theta_j(S) = \theta(\operatorname{tr}\sigma_j S) \qquad (j = 1, 2).$$

Since we are interested in the θ_j-Fourier coefficients of Siegel cusp forms, we can in fact assume that the σ_j belong to the set

$$\mathcal{R}_n^+ = \text{set of half-integral positive definite symmetric matrices } \sigma \in \mathrm{GL}_n(\mathbf{Q})$$

(this means 2σ has integer entries and even diagonal entries).

6.2. **The setup and the spectral side.** Given a continuous function φ on $\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})$, its Whittaker function along the unipotent subgroup $N = \{\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix} \mid S \in S_n(\mathbf{A})\}$ is defined by

$$W_\varphi(g,\chi) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \varphi(ng)\overline{\chi(n)}dn$$

for $g \in \mathbb{G}(\mathbf{A})$ and χ a character. Write $n_S = \left(\begin{smallmatrix} I_n & S \\ O & I_n \end{smallmatrix}\right)$ for $S \in S_n(\mathbf{A})$. There exits $\sigma \in S_n(\mathbf{Q})$ such that $\chi(n_S) = \theta(\operatorname{tr}\sigma S)$ for all S. Define

$$c_\sigma(\varphi) = W(1,\chi) = \int_{S_n(\mathbf{Q})\backslash S_n(\mathbf{A})} \varphi(n_S)\overline{\theta(\operatorname{tr}\sigma S)}dS. \tag{6.2}$$

In §5.1, we defined a test function, henceforth to be denoted f_N, from the following data: a finite set $\mathbf{S}$ of primes, a level N coprime to $\mathbf{S}$, a compact set $C_p \subseteq G(\mathbf{Q}_p)$ for each $p \in \mathbf{S}$, and a weight $\mathtt{k} > 2n$. In this section, we compute the following limit:

$$I = \lim_{\substack{N\to\infty \\ (N,\mathbf{S})=1}} \iint_{(N(\mathbf{Q})\backslash N(\mathbf{A}))^2} \frac{K_{f_N}(n_1,n_2)}{\psi(N)}\overline{\theta_1(n_1)}\theta_2(n_2)dn_1dn_2, \tag{6.3}$$

where $\psi(N) = \text{meas}(\overline{K_0(N)})^{-1}$. We will compute I in two ways, using the spectral and geometric forms of the kernel function. The preliminary form of the resulting formula is given in Theorem 6.3 below.

Using the spectral form (5.6) of the kernel function along with (6.2), we immediately obtain

$$I = \lim_{\substack{N\to\infty \\ (N,\mathbf{S})=1}} \frac{1}{\psi(N)} \sum_{\pi\in\Pi_{\mathbf{k}}(N)} \sum_{\varphi\in E_{\mathbf{k}}(\pi,N)} \frac{c_{\sigma_1}(\varphi)\overline{c_{\sigma_2}(\varphi)}}{\|\varphi\|^2} \prod_{p\in S}(\mathcal{S}f_p)(t_{\pi_p}). \tag{6.4}$$

6.3. **The geometric side.** First we show how the limit can be eliminated on the geometric side.

Proposition 6.1. *Let $f = f_1$ be the test function we have defined when $N = 1$. (Its finite part is the characteristic function of $\prod Z_pC_p$, where $C_p = K_p$ for all $p \notin \mathbf{S}$.) Then for I as in* (6.3),

$$I = \iint_{(N(\mathbf{Q})\backslash N(\mathbf{A}))^2} \sum_{\gamma\in\mathbb{P}(\mathbf{Q})} f(n_1^{-1}\gamma n_2)\overline{\theta_1(n_1)}\theta_2(n_2)dn_1dn_2. \tag{6.5}$$

Proof. Recall that $[0,1)\times\widehat{\mathbf{Z}}$ is a fundamental domain in $\mathbf{A}$ for $\mathbf{Q}\backslash\mathbf{A}$. We may therefore replace $N(\mathbf{Q})\backslash N(\mathbf{A}) \cong N(\mathbf{Q}\backslash\mathbf{A})$ by the compact set $J = N([0,1]\times\widehat{\mathbf{Z}})$. Applying Proposition 5.4 with $J_1 = J_2 = J$, the integrand in (6.3) is absolutely bounded by the constant $\sum\alpha_\gamma$. Hence by the dominated convergence theorem,

$$\begin{aligned} I &= \lim_{\substack{N\to\infty \\ (N,\mathbf{S})=1}} \iint_{J\times J} \frac{K_{f_N}(n_1,n_2)}{\psi(N)}\overline{\theta_1(n_1)}\theta_2(n_2)dn_1dn_2 \\ &= \iint_{J\times J} \sum_{\gamma\in\mathbb{G}(\mathbf{Q})} \lim_{\substack{N\to\infty \\ (N,\mathbf{S})=1}} \frac{f_N(n_1^{-1}\gamma n_2)}{\psi(N)}\overline{\theta_1(n_1)}\theta_2(n_2)dn_1dn_2. \end{aligned}$$

By Proposition 5.1, we can assume $r(\gamma) = \pm r$. Furthermore, if

$$f_{N,\text{fin}}(n_{1\,\text{fin}}^{-1}\gamma n_{2\,\text{fin}}) \neq 0$$

for $n_{j\,\text{fin}} \in N(\widehat{\mathbf{Z}})$, then

$$n_{1\,\text{fin}}^{-1}\gamma n_{2\,\text{fin}} \in \prod_{p\in\mathbf{S}} C_p \prod_{p\notin\mathbf{S}} K_0(N)_p.$$

Therefore

$$\gamma = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \in \prod_{p\in\mathbf{S}} C_p \prod_{p\notin\mathbf{S}} K_0(N)_p \subseteq M_{2n}(\widehat{\mathbf{Z}}).$$

In particular $Y \in M_n(N\mathbf{Z})$. Hence for any $Y \neq 0$, $f_{N,\text{fin}}(n_{1\,\text{fin}}^{-1}\gamma n_{2\,\text{fin}}) = 0$ when N is sufficiently large. On the other hand, if $Y = 0$, then for $n_j \in J$,

$$\frac{f_N(n_1^{-1}\gamma n_2)}{\psi(N)} = f(n_1^{-1}\gamma n_2) = f_\infty(n_{1,\infty}^{-1}\gamma n_{2,\infty})f_{\text{fin}}(\gamma)$$

for $f = f_1$, which is obviously independent of N. In particular,

$$I = \iint_{J\times J} \sum_{\gamma\in\mathbb{P}(\mathbf{Q})} f_1(n_1^{-1}\gamma n_2)\overline{\theta_1(n_1)}\theta_2(n_2)dn_1dn_2.$$

As a function of n_1 and n_2, the summation over $\mathbb{P}(\mathbf{Q})$ is $N(\mathbf{Q})$-invariant in both variables. So we can replace the region of the double integral by $(N(\mathbf{Q})\backslash N(\mathbf{A}))^2$. This completes the proof. □

We now examine the sum in (6.5). By Proposition 5.1, we may assume that $r(\gamma) = \pm r$. Because $f_\infty(g) = 0$ if $r(g) < 0$, we may in fact take $r(\gamma) = r$. By a variant of (3.4), we may write

$$\gamma = \eta\, g_{A,r},$$

where $\eta \in N(\mathbf{Q})$ and, for $A \in \mathrm{GL}_n(\mathbf{Q})$,

$$g_{A,r} := \begin{pmatrix} A & \\ & r\,{}^tA^{-1} \end{pmatrix}.$$

Recalling that $\gamma \in \overline{P}(\mathbf{Q})$ is only defined up to multiplication by the center, we observe that for $\lambda \in \mathbf{Q}^*$, $\lambda g_{A,r} = g_{\lambda A, r\lambda^2}$. Hence by our insistance that $r(\gamma) = r$, we may only scale by $\lambda = \pm 1$. Therefore

$$\sum_{\gamma \in \overline{P}(\mathbf{Q})} f(n_1^{-1}\gamma n_2) = \sum_{A \in \mathrm{GL}_n(\mathbf{Q})/\{\pm I_n\}} \sum_{\eta \in N(\mathbf{Q})} f(n_1^{-1}\eta g_{A,r} n_2).$$

Taking, as we may, n_1, n_2 to range through the fundamental domain $N([0,1] \times \widehat{\mathbf{Z}})$ for $N(\mathbf{Q})\backslash N(\mathbf{A})$, by (5.3) a given summand vanishes unless

$$\eta g_{A,r} \in n_{1\,\mathrm{fin}} M_{2n}(\widehat{\mathbf{Z}}) n_{2\,\mathrm{fin}}^{-1} \subseteq M_{2n}(\widehat{\mathbf{Z}}).$$

This implies that $A \in M_n(\mathbf{Z})$ and $rA^{-1} \in M_n(\mathbf{Z})$. Hence

$$I = \sum_{\substack{A \in \{\pm I_n\}\backslash M_n(\mathbf{Z}), \\ rA^{-1} \in M_n(\mathbf{Z})}} \iint_{(S_n(\mathbf{Q})\backslash S_n(\mathbf{A}))^2} \sum_{S \in S_n(\mathbf{Q})} f\left(\begin{pmatrix} I_n & -S_1 \\ O & I_n \end{pmatrix}\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix} g_{A,r} \begin{pmatrix} I_n & S_2 \\ O & I_n \end{pmatrix}\right)$$

$$\times \theta(-\operatorname{tr}\sigma_1 S_1 + \operatorname{tr}\sigma_2 S_2)\, dS_1 dS_2.$$

The double integral becomes

$$\int_{S_n(\mathbf{Q})\backslash S_n(\mathbf{A})} \int_{S_n(\mathbf{A})} f\left(\begin{pmatrix} I_n & -S_1 \\ O & I_n \end{pmatrix} g_{A,r} \begin{pmatrix} I_n & S_2 \\ O & I_n \end{pmatrix}\right) \theta(-\operatorname{tr}\sigma_1 S_1 + \operatorname{tr}\sigma_2 S_2)\, dS_1 dS_2$$

$$= \int_{S_n(\mathbf{Q})\backslash S_n(\mathbf{A})} \int_{S_n(\mathbf{A})} f\left(\begin{pmatrix} I_n & -(S_1 - r^{-1}AS_2\,{}^tA) \\ O & I_n \end{pmatrix} g_{A,r}\right) \theta(-\operatorname{tr}\sigma_1 S_1 + \operatorname{tr}\sigma_2 S_2)\, dS_1 dS_2.$$

Making the substitution $S_1' = S_1 - r^{-1}AS_2\,{}^tA$, the above is

$$= \iint_{S_n(\mathbf{A}) \times (S_n(\mathbf{Q})\backslash S_n(\mathbf{A}))} f(\left(\begin{pmatrix} I_n & -S_1' \\ O & I_n \end{pmatrix} g_{A,r}\right) \theta(-\operatorname{tr}(\sigma_1 S_1' + r^{-1}\sigma_1 AS_2\,{}^tA) + \operatorname{tr}\sigma_2 S_2)\, dS_1' dS_2$$

$$= \int_{S_n(\mathbf{A})} f(\left(\begin{pmatrix} I_n & -S_1' \\ O & I_n \end{pmatrix} g_{A,r}\right) \theta(-\operatorname{tr}\sigma_1 S_1')\, dS_1' \int_{S_n(\mathbf{Q})\backslash S_n(\mathbf{A})} \theta(\operatorname{tr}(\sigma_2 S_2 - r^{-1}\sigma_1 AS_2\,{}^tA))\, dS_2.$$

The value of the second integral is

$$\begin{cases} \mathrm{meas}(S_n(\mathbf{Q})\backslash S_n(\mathbf{A})) = 1 & \text{if } \theta(\operatorname{tr}(\sigma_2 S_2 - r^{-1}\sigma_1 AS_2\,{}^tA)) = 1 \text{ for all } S_2 \in S_n(\mathbf{A}), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6.2. *We have* $\theta(\operatorname{tr}(\sigma_2 S - r^{-1}\sigma_1 AS\,{}^tA)) = 1$ *for all* $S \in S_n(\mathbf{A})$ *if and only if* ${}^tA\sigma_1 A = r\sigma_2$.

Proof. By the fact that $\operatorname{tr} AB = \operatorname{tr} BA$,

$$\theta(\operatorname{tr}(\sigma_2 S - r^{-1}\sigma_1 AS\,{}^tA)) = \theta(\operatorname{tr}((\sigma_2 - r^{-1}\,{}^tA\sigma_1 A)S)).$$

The lemma follows from this. □

6.4. **Main formula.** The result of the above computation is the following asymptotic Petersson formula for PGSp($2n$).

Theorem 6.3. *Let* $\mathbf{S}$ *be a finite set of prime numbers, fix sets* C_p *as in §5.1, and let* $r \geq 1$ *be the integer defined in* (5.1). *Let* $f = f_1$ *be the associated test function defined there for the case of level* $N = 1$. *Then for* $\mathtt{k} > 2n$ *with* $2|n\mathtt{k}$, *and any* $\sigma_1, \sigma_2 \in \mathcal{R}_n^+$,

$$\lim_{\substack{N\to\infty \\ (N,\mathbf{S})=1}} \frac{1}{\psi(N)} \sum_{\pi\in\Pi_{\mathtt{k}}(N)} \left(\sum_{\varphi\in E_{\mathtt{k}}(\pi,N)} \frac{c_{\sigma_1}(\varphi)\overline{c_{\sigma_2}(\varphi)}}{\|\varphi\|^2} \right) \prod_{p\in\mathbf{S}} (\mathcal{S}f_p)(t_{\pi_p})$$

$$(6.6) \qquad = \sum_A \int_{S_n(\mathbf{A})} f(\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix} \begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix}) \theta(\mathrm{tr}\,\sigma_1 S) dS.$$

Here, $S_n(\mathbf{A})$ *is the set of symmetric* $n\times n$ *matrices over* $\mathbf{A}$, *and* A *runs through the finite set of rank* n *matrices in* $M_n(\mathbf{Z})/\{\pm I_n\}$ *satisfying the following conditions:*

(1) $r\,{}^tA^{-1} \in M_n(\mathbf{Z})$.
(2) ${}^tA\sigma_1 A = r\sigma_2$.

Remarks: (1) In Appendix B, we give a quantitative version of the above for GSp(4).

(2) Haar measure is normalized as follows. On the geometric side and in the definition of the Fourier coefficients (6.2) we take $\mathrm{meas}(S_n(\widehat{\mathbf{Z}})) = 1$ and Lebesgue measure on the Euclidean space $S_n(\mathbf{R})$. The Satake transform is defined using $\mathrm{meas}(K_p) = 1$. The archimedean test function f_∞ depends (via $d_{\mathtt{k}}$) on an unspecified choice of measure on $\mathbb{G}(\mathbf{R})$. This choice materializes on the spectral side in $\|\varphi\|^{-2}$. The exact relationship between several natural choices of Haar measure on $\mathrm{Sp}_{2n}(\mathbf{R})$ is computed in [PSS, §A].

Proof. In view of the discussion in the previous two subsections, it just remains to prove that only finitely many matrices A satisfy the given conditions. For fixed j, let $d \in \mathbf{R}$ be the entry in the j-th row and j-th column of $r\sigma_2$. Suppose $A \in M_n(\mathbf{Z})$ satisfies condition 2, and let $v \in \mathbf{Z}^n$ denote the j-th column of A. Then ${}^tv\sigma_1 v = d$. We will show that there are only finitely many such v (this is well-known), from which it follows that the set of A is also finite since j is arbitrary.

Because σ_1 is symmetric, there exists an orthogonal matrix Q such that $\sigma_1 = {}^tQ\Lambda Q$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is diagonal. Furthermore the eigenvalues λ_j are all positive since σ_1 is positive definite. It follows that the linear map $Q : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ restricts to give an isometry between the sets

$$X = \{v \in \mathbf{R}^n |\, {}^tv\sigma_1 v = d\}$$

and

$$Y = \{w \in \mathbf{R}^n |\, {}^tw\Lambda w = d\}.$$

Notice that Y is the ellipsoid

$$\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 = d,$$

which is compact. Hence X is also compact. Since $\mathbf{Z}^n$ is discrete, it follows that there are only finitely many integer lattice points in X, as claimed. □

7. Refinement of the geometric side

Fix a matrix $A \in M_n(\mathbf{Z})$ satisfying the hypotheses of Theorem 6.3. The integral on the geometric side of (6.6) can be factorized as

$$I_A = \prod_{p\leq\infty} I_{A,p} = \prod_{p\leq\infty} \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix} \begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix}) \theta_p(\mathrm{tr}\,\sigma_1 S) dS.$$

We will see that for all $p \notin \mathbf{S}$, $I_{A,p}$ can be computed explicitly, and is independent of A when $\mathtt{k}$ is even.

7.1. **Archimedean integral.** When $p = \infty$, by Corollary A.10 in the Appendix,

$$I_{A,\infty} = \int_{S_n(\mathbf{R})} f_\infty(\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix} \begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix}) e^{2\pi i \operatorname{tr} \sigma_1 S} dS$$

$$= \int_{S_n(\mathbf{R})} \frac{d_{\mathtt{k}} 2^{n\mathtt{k}} r^{n\mathtt{k}/2} e^{2\pi i \operatorname{tr} \sigma_1 S}}{\det(A + r\,{}^tA^{-1} + iSr\,{}^tA^{-1})^{\mathtt{k}}}\, dS$$

$$(7.1) \qquad = \frac{d_{\mathtt{k}} 2^{n\mathtt{k}} (\det A)^{\mathtt{k}}}{r^{n\mathtt{k}/2}} \int_{S_n(\mathbf{R})} \frac{e^{2\pi i \operatorname{tr} \sigma_1 S}}{\det(r^{-1} A\,{}^tA + I_n + iS)^{\mathtt{k}}}\, dS.$$

We apply the following formula of Siegel. Let

$$(7.2) \qquad \Gamma_n(a) = \pi^{n(n-1)/4} \prod_{j=1}^{n} \Gamma(a - \tfrac{1}{2}(j-1)).$$

Then for $\delta > \frac{n-1}{2}$ and symmetric matrices $X_0, \Lambda > 0$,

$$\int_{S_n(\mathbf{R})} \frac{e^{\operatorname{tr} i\Lambda Y}}{\det(X_0 + iY)^{\delta + (n+1)/2}} dY = \frac{1}{i^{n(n+1)/2} 2^{(n-1)n/2}} \frac{(2\pi i)^{n(n+1)/2} (\det \Lambda)^{\delta}}{\Gamma_n(\delta + (n+1)/2)} e^{-\operatorname{tr} \Lambda X_0}$$

$$(7.3) \qquad = \frac{2^n \pi^{n(n+1)/2} (\det \Lambda)^{\delta}}{\Gamma_n(\delta + (n+1)/2)} e^{-\operatorname{tr} \Lambda X_0}$$

(cf. [Si1, Hilfssatz 37, p. 585]; we have used the form given by Herz [H, (1.2)]). In [H], the measure is $dZ = \prod_{j \le k} dz_{jk}$, where $Z = X_0 + iY = (\eta_{jk} z_{jk})$ with $\eta_{jk} = 1$ if $j = k$, and $1/2$ otherwise. Thus

$$dZ = i^{n(n+1)/2} 2^{(n-1)n/2} dY,$$

which explains the first factor in the above formula.

We evaluate (7.1) using (7.3) with $\delta = \mathtt{k} - (n+1)/2 > (n-1)/2$, $X_0 = I_n + r^{-1} A\,{}^tA$, and $\Lambda = 2\pi\sigma_1$. By condition (2) of Theorem 6.3, $\det(A) = \pm r^{n/2} \left(\frac{\det \sigma_2}{\det \sigma_1}\right)^{1/2}$. So (7.1) becomes

$$\frac{d_{\mathtt{k}} 2^{n\mathtt{k}}}{r^{n\mathtt{k}/2}} \operatorname{sgn}(\det A)^{\mathtt{k}} r^{n\mathtt{k}/2} \left(\frac{\det \sigma_2}{\det \sigma_1}\right)^{\mathtt{k}/2} 2^n \pi^{n(n+1)/2} \frac{\det(2\pi\sigma_1)^{\mathtt{k} - (n+1)/2}}{\Gamma_n(\mathtt{k})} e^{-2\pi \operatorname{tr}(\sigma_1 (I_n + r^{-1} A\,{}^tA))}.$$

We simplify the above using $\operatorname{tr}(\sigma_1 r^{-1} A\,{}^tA) = \operatorname{tr}(r^{-1}\,{}^tA \sigma_1 A) = \operatorname{tr} \sigma_2$ for matrices A as in Theorem 6.3. The result is the following:

$$(7.4) \qquad I_{A,\infty} = \operatorname{sgn}(\det A)^{\mathtt{k}}\, d_{\mathtt{k}} \left(\frac{\det \sigma_2}{\det \sigma_1}\right)^{\mathtt{k}/2} \frac{(\det \sigma_1)^{\mathtt{k} - (n+1)/2} (4\pi)^{n\mathtt{k}}}{2^{n(n-1)/2} \Gamma_n(\mathtt{k})\, e^{2\pi \operatorname{tr}(\sigma_1 + \sigma_2)}}.$$

Observe that this is independent of A when $\mathtt{k}$ is even.

7.2. **Nonarchimedean integrals:** $p \notin \mathbf{S}$. In this case, $r \in \mathbf{Z}_p^*$, and $A \in \mathrm{GL}_n(\mathbf{Z}_p)$. It follows that

$$f_p(\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix}\begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix}) \neq 0 \iff \begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix}\begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix} \in K_p$$
$$\iff S \in M_n(\mathbf{Z}_p).$$

For such S, $\theta_p(\mathrm{tr}\,\sigma_1 S) = 1$, so we find that

$$I_{A,p} = \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix}\begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix})\theta_p(\mathrm{tr}\,\sigma_1 S)dS = \mathrm{meas}(S_n(\mathbf{Z}_p)) = 1.$$

(Recall that we use the $N = 1$ test function f_1 in (6.6), so that f_p is the characteristic function of Z_pK_p when $p \notin \mathbf{S}$.)

7.3. **Proof of Theorem 1.1.** When $\mathbf{S} = \emptyset$, we now have a completely explicit expression for the right-hand side of (6.6). Since $r = 1$, the sum over A is nonzero only if ${}^tA\sigma_1 A = \sigma_2$ for some $A \in \mathrm{GL}_n(\mathbf{Z})/\{\pm I_n\}$. In particular, $(\det A)^2 = 1$, so $\det\sigma_1 = \det\sigma_2$. If we let I_A denote the summand indexed by A in Theorem 6.3, then by the above discussion,

$$I_A = (\det A)^{\mathtt{k}}\, d_{\mathtt{k}}\, (4\pi)^{n\mathtt{k}-n(n-1)/4} \frac{(\det\sigma_1)^{\mathtt{k}-(n+1)/2}}{\prod_{j=1}^n \Gamma(\mathtt{k} - \frac{j-1}{2})} e^{-2\pi\,\mathrm{tr}(\sigma_1+\sigma_2)}. \tag{7.5}$$

We wish to express the spectral side in classical terms. Let $S_{\mathtt{k}}(N)$ be the space of Siegel cusp forms F satisfying

$$F(\gamma\cdot Z) = j(\gamma, Z)^{\mathtt{k}} F(Z)$$

for all $Z \in \mathcal{H}_n$ and

$$\gamma \in \Gamma_0(N) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbf{Z}) |\, C \equiv O \mod N\},$$

where

$$j(g, Z) = r(g)^{-n/2}\det(CZ + D) \qquad (g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G(\mathbf{R})).$$

Any $F \in S_{\mathtt{k}}(N)$ has a Fourier expansion

$$F(Z) = \sum_{\sigma\in\mathcal{R}_n^+} a_\sigma(F) e^{2\pi i\,\mathrm{tr}\,\sigma Z}, \quad (Z \in \mathcal{H}_n). \tag{7.6}$$

We normalize the Petersson/Maass scalar product on $S_{\mathtt{k}}(N)$ by

$$\langle F, H\rangle = \frac{1}{\psi(N)} \int_{\Gamma_0(N)\backslash\mathcal{H}_n} F(Z)\overline{H(Z)}(\det Y)^{\mathtt{k}-n-1} dX\, dY \quad (Z = X + iY), \tag{7.7}$$

where $\psi(N) = [K_{\mathrm{fin}} : K_0(N)] = [\Gamma_0(1) : \Gamma_0(N)]$.

We need to choose the quotient measure on $\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})$ compatibly with the above. For any $N \geq 1$, let $D_N \subseteq \mathcal{H}_n$ be a fundamental domain for $\Gamma_0(N)\backslash\mathcal{H}_n$, identified with a subset of $\mathrm{Sp}_{2n}(\mathbf{R})$ via

$$Z = X + iY \leftrightarrow g_Z = \begin{pmatrix} Y^{1/2} & XY^{-1/2} \\ O & Y^{-1/2} \end{pmatrix}.$$

Then, as in [KL1, Prop. 7.43] for example, we may define a quotient measure on $\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})$ by

$$\int_{\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})} h(g)dg = \int_{D_N K_\infty \times K_0(N)} h(g_Z k_\infty \times k_{\mathrm{fin}}) \frac{dX\, dY}{(\det Y)^{n+1}} dk_\infty dk_{\mathrm{fin}},$$

where the compact groups K_∞ and K_{fin} each have total volume 1. This measure dg is independent of the choice of N since

$$\mathrm{meas}(D_N)\,\mathrm{meas}(K_0(N)) = [\overline{\Gamma_0(1)} : \overline{\Gamma_0(N)}]\,\mathrm{meas}(D_1)\psi(N)^{-1} = \mathrm{meas}(D_1).$$

Taking $N=1$, a well-known computation of Siegel ([Si2]) gives

$$\mathrm{meas}(\mathbb{G}(\mathbf{Q})\backslash\mathbb{G}(\mathbf{A})) = \mathrm{meas}(\mathrm{Sp}_{2n}(\mathbf{Z})\backslash\mathcal{H}_n) = 2\prod_{j=1}^{n}[(j-1)!\pi^{-j}\zeta(2j)].$$

Given $F \in S_{\mathrm{k}}(N)$, its adelic counterpart is the function $\varphi_F \in L_0^2$ defined by

$$\varphi_F(g) = F(g_\infty \cdot iI_n)j(g_\infty, iI_n)^{-\mathrm{k}} \tag{7.8}$$

for $g = \gamma(g_\infty \times k) \in G(\mathbf{A}) = G(\mathbf{Q})(G(\mathbf{R})^+ \times K_0(N))$. The well-definedness of φ_F is a consequence of the fact that $G(\mathbf{Q})\cap(G(\mathbf{R})^+\times K_0(N)) = \Gamma_0(N)$. With measures normalized as above, the map $F \mapsto \varphi_F$ defines a linear isometry from $S_{\mathrm{k}}(N)$ onto the subspace $A_{\mathrm{k}}(N) \subseteq L_0^2$ defined in (4.2). This may be proven just as in [Sa]. (The latter paper works with the principal congruence subgroup of level N, but the Siegel parabolic case is just the same.) The relationship between the adelic and classical Fourier coefficients is given by the following (see (6.2)).

Proposition 7.1. *For $F \in S_{\mathrm{k}}(N)$ and $\sigma \in S_n(\mathbf{Q})$,*

$$c_{\varphi_F}(\sigma) = \begin{cases} e^{-2\pi\,\mathrm{tr}\,\sigma} a_F(\sigma) & \text{if } \sigma \in \mathcal{R}_n^+ \\ 0 & \text{otherwise.}\end{cases} \tag{7.9}$$

Proof. By strong approximation for the adeles,

$$S_n(\mathbf{A}) = S_n(\mathbf{Q}) + S_n(\mathbf{R}) \times S_n(\widehat{\mathbf{Z}}).$$

Decomposing $S = X_{\mathbf{Q}} + (X_\infty \times X_{\mathrm{fin}}) \in S_n(\mathbf{A})$ accordingly, by the right $K_0(N)$-invariance and left $G(\mathbf{Q})$-invariance of φ_F we have

$$\varphi_F(\begin{pmatrix} I_n & S \\ & I_n\end{pmatrix}) = \varphi_F(\begin{pmatrix} I_n & X_\infty \\ & I_n\end{pmatrix}).$$

Also, it follows that

$$S_n(\mathbf{Q})\backslash S_n(\mathbf{A}) = S_n(\mathbf{Z})\backslash(S_n(\mathbf{R}) \times S_n(\widehat{\mathbf{Z}})).$$

If D is any fundamental domain for $S_n(\mathbf{Z})\backslash S_n(\mathbf{R})$, then $D \times S_n(\widehat{\mathbf{Z}})$ is a fundamental domain for $S_n(\mathbf{Z})\backslash(S_n(\mathbf{R}) \times S_n(\widehat{\mathbf{Z}}))$ ([KL1, Theorem 7.40]). Therefore (6.2) becomes

$$c_{\varphi_F}(\sigma) = \int_{S_n(\mathbf{Z})\backslash S_n(\mathbf{R})} \varphi_F(\begin{pmatrix} I_n & X_\infty \\ & I_n\end{pmatrix})\theta_\infty(-\mathrm{tr}(\sigma X_\infty))dX_\infty \int_{S_n(\widehat{\mathbf{Z}})} \theta_{\mathrm{fin}}(-\mathrm{tr}(\sigma X_{\mathrm{fin}}))dX_{\mathrm{fin}}.$$

Write $X_{\mathrm{fin}} = (X_{ij})$ for $X_{ij} \in \mathbf{A}_{\mathrm{fin}}$. If we write $\sigma = (b_{ij}\sigma_{ij})$, where b_{ij} is equal to 1 or 1/2 according to whether or not $i=j$, then $\mathrm{tr}(\sigma X_{\mathrm{fin}}) = \sum_{i,j}\sigma_{ij}X_{ij}$. Note that $\sigma_{ij} \in \mathbf{Z}$ for all i,j if and only if σ is half-integral, i.e., $\sigma \in \mathcal{R}_n$. If this condition does not hold, then the second integral vanishes.

Assuming $\sigma \in \mathcal{R}_n$, using (6.1) and (7.8) we have

$$c_{\varphi_F}(\sigma) = \int_{S_n(\mathbf{Z})\backslash S_n(\mathbf{R})} F(X+iI_n)e^{-2\pi i\,\mathrm{tr}(\sigma X)}dX$$

$$= e^{-2\pi\,\mathrm{tr}\,\sigma}\int_{S_n(\mathbf{Z})\backslash S_n(\mathbf{R})} F(X+iI_n)e^{-2\pi i\,\mathrm{tr}(\sigma(X+iI_n))}dX$$

$$= e^{-2\pi\,\mathrm{tr}\,\sigma}a_F(\sigma).$$

□

Letting $\mathcal{B}_{\mathtt{k}}(N)$ be an orthogonal basis for $S_{\mathtt{k}}(N)$, in the special case $S = \emptyset$ and $r = 1$, our main formula (6.3) becomes

$$\lim_{N\to\infty} \frac{1}{\psi(N)} \sum_{F\in\mathcal{B}_{\mathtt{k}}(N)} \frac{a_{\sigma_1}(F)\overline{a_{\sigma_2}(F)}}{\|F\|^2} = \sum_{\substack{A\in \mathrm{GL}_n(\mathbf{Z})/\pm I_n\\ {}^tA\sigma_1 A=\sigma_2}} (\det A)^{\mathtt{k}}\, d_{\mathtt{k}}\, (4\pi)^{n\mathtt{k}-n(n-1)/4} \frac{(\det\sigma_1)^{\mathtt{k}-(n+1)/2}}{\prod_{j=1}^n \Gamma(\mathtt{k}-\frac{j-1}{2})}. \tag{7.10}$$

For the formal degree, we will take the classical measure on $\mathrm{Sp}(2n)$ corresponding to the measure in (7.7). So as shown in [PSS],

$$d_{\mathtt{k}} = \frac{1}{2^n(4\pi)^{n(n+1)/2}} \prod_{1\le i\le j\le n} (2\mathtt{k}-(i+j)).$$

(See the final remark after Proposition A.7 below.) We can simplify $\frac{d_{\mathtt{k}}}{\Gamma_n(\mathtt{k})}$ using the following.

Lemma 7.2. *For any integers $n \ge 1$ and $\mathtt{k} > 2n$,*

$$\frac{\prod_{1\le i\le j\le n}(2\mathtt{k}-(i+j))}{\prod_{\ell=1}^n \Gamma(\mathtt{k}-\frac{\ell-1}{2})} = \frac{2^{n(n+1)/2}}{\prod_{j=1}^n \Gamma(\mathtt{k}-\frac{n+j}{2})}.$$

Proof. When $n = 1$, by the functional equation for the Gamma function we have

$$\frac{(2\mathtt{k}-2)}{\Gamma(\mathtt{k})} = \frac{2(\mathtt{k}-1)}{\Gamma(\mathtt{k})} = \frac{2}{\Gamma(\mathtt{k}-1)},$$

as needed. Given $n \ge 2$, suppose the formula holds for $n-1$. Then

$$\begin{aligned}\frac{\prod_{1\le i\le j\le n}(2\mathtt{k}-(i+j))}{\prod_{\ell=1}^n \Gamma(\mathtt{k}-\frac{\ell-1}{2})} &= \frac{\prod_{1\le i\le j\le n-1}(2\mathtt{k}-(i+j))}{\prod_{\ell=1}^{n-1} \Gamma(\mathtt{k}-\frac{\ell-1}{2})}\cdot\frac{\prod_{1\le i\le n}(2\mathtt{k}-(i+n))}{\Gamma(\mathtt{k}-\frac{n-1}{2})}\\ &= \frac{2^{n(n-1)/2}}{\prod_{j=1}^{n-1}\Gamma(\mathtt{k}-\frac{n-1+j}{2})}\cdot\frac{\prod_{1\le i\le n}(2\mathtt{k}-(i+n))}{\Gamma(\mathtt{k}-\frac{n-1}{2})}\\ &= 2^{n(n+1)/2}\prod_{i=1}^n \frac{(\mathtt{k}-\frac{i+n}{2})}{\Gamma(\mathtt{k}-\frac{i+n}{2}+1)} = \frac{2^{n(n+1)/2}}{\prod_{i=1}^n\Gamma(\mathtt{k}-\frac{n+i}{2})}.\end{aligned}$$

□

It now follows that

$$\lim_{N\to\infty}\frac{1}{\psi(N)}\sum_{F\in\mathcal{B}_{\mathtt{k}}(N)}\frac{a_{\sigma_1}(F)\overline{a_{\sigma_2}(F)}}{\|F\|^2} = \frac{(\det\sigma_1)^{\mathtt{k}-(n+1)/2}}{\pi^{n(n-1)/4}(4\pi)^{n(n+1)/2-n\mathtt{k}}\prod_{j=1}^n\Gamma(\mathtt{k}-\frac{n+j}{2})}\delta_{\mathtt{k}}(\sigma_1,\sigma_2), \tag{7.11}$$

for $\delta_{\mathtt{k}}(\sigma_1,\sigma_2)$ given in (1.3). This proves Theorem 1.1.

7.4. **Nonarchimedean integrals:** $p \in \mathbf{S}$**.** This case is more difficult. Our goal is to compute (or bound) the local integral

$$I_{A,p} = I_{A,p}(f_p) = \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} I_n & S\\ O & I_n\end{pmatrix}\begin{pmatrix} A & O\\ O & r\,{}^tA^{-1}\end{pmatrix})\theta_p(\mathrm{tr}\,\sigma_1 S)dS.$$

To simplify the computation, we may essentially reduce to the case where A is diagonal, as follows. By the elementary divisors theorem, there exist $U, V \in \mathrm{GL}_n(\mathbf{Z})$ and a diagonal matrix

$$D = \mathrm{diag}(d_1, \dots, d_n)$$

with positive integer entries satisfying $d_1|d_2|\cdots|d_n$, such that

$$A = UDV. \tag{7.12}$$

Proposition 7.3. *For A as above,*

$$I_{A,p} = \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix} \begin{pmatrix} D & O \\ O & rD^{-1} \end{pmatrix})\theta_p(\operatorname{tr} \sigma_1' S) dS,$$

where

$$\sigma_1' = \sigma_U = {}^tU \sigma_1 U.$$

Proof. By definition,

$$\begin{aligned} I_{A,p} &= \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} I_n & S \\ O & I_n \end{pmatrix} \begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix})\theta_p(\operatorname{tr} \sigma_1 S) dS \\ &= \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} UDV & rS\,{}^tU^{-1}D^{-1}\,{}^tV^{-1} \\ O & r\,{}^tU^{-1}D^{-1}\,{}^tV^{-1} \end{pmatrix})\theta_p(\operatorname{tr} \sigma_1 S) dS. \end{aligned}$$

Because f_p is bi-K_p-invariant, we are free to multiply its argument on the left by $\begin{pmatrix} U^{-1} & \\ & {}^tU \end{pmatrix} \in K_p$ and on the right by $\begin{pmatrix} V^{-1} & \\ & {}^tV \end{pmatrix} \in K_p$. This gives

$$I_{A,p} = \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} D & r(U^{-1}S\,{}^tU^{-1})D^{-1} \\ O & rD^{-1} \end{pmatrix})\theta_p(\operatorname{tr} \sigma_1 S) dS.$$

Let $S' = U^{-1}S\,{}^tU^{-1}$. Then $dS' = dS$ since $S \mapsto S'$ is an isomorphism mapping $S(\widehat{\mathbf{Z}})$ to $S(\widehat{\mathbf{Z}})$. Hence the above is

$$= \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} D & rS'D^{-1} \\ O & rD^{-1} \end{pmatrix})\theta_p(\operatorname{tr} \sigma_1 US'\,{}^tU) dS'.$$

Now using $\operatorname{tr} \sigma_1 US\,{}^tU = \operatorname{tr} {}^tU\sigma_1 US$, we find

$$I_{A,p} = \int_{S_n(\mathbf{Q}_p)} f_p(\begin{pmatrix} D & rSD^{-1} \\ O & rD^{-1} \end{pmatrix})\theta_p(\operatorname{tr} \sigma_1' S) dS. \qquad \square$$

For the purpose of computing the above local integral, by the K_p-invariance of f_p, we may assume that each diagonal entry of D is a power of p. Thus, we take

$$D = \operatorname{diag}(p^{\alpha_1}, \ldots, p^{\alpha_n}),$$

for

$$0 \le \alpha_1 \le \cdots \le \alpha_n.$$

For $x \in \mathbf{Q}_p$, we define $\operatorname{ord}_p(x) = n$ if $x = p^n z$ with z a unit. So $r_p = \operatorname{ord}_p(r)$, for example.

Proposition 7.4. *With the above notation, suppose $I_{A,p} \neq 0$. Then each $\alpha_j \le r_p$. Under the additional assumption that $\operatorname{ord}_p(\det \sigma_1) = \operatorname{ord}_p(\det \sigma_2)$, we further have*

$$\alpha_1 + \cdots + \alpha_n = \frac{nr_p}{2}.$$

Proof. As in Theorem 6.3, we are assuming that $r\,{}^tA^{-1} \in M_n(\mathbf{Z})$. It follows that likewise $rD^{-1} \in M_n(\mathbf{Z})$, and hence $\alpha_j \le r_p$ for each j. Under the additional assumption, taking determinants in the relation ${}^tA\sigma_1 A = r\sigma_2$ gives $p^{2(\alpha_1+\cdots+\alpha_n)} = p^{nr_p}$, and the last assertion follows. $\square$

In principle, one can now compute the integral by applying Proposition 3.2 and considering various cases to obtain certain exponential sums. We will discuss this process in more detail for the special case of GSp(4) in Section 9. It should be evident from this special case that the general case is very complicated.

We conclude the present section by giving a trivial bound for $I_{A,p}$.

Proposition 7.5. *With notation as above,*

$$|I_{A,p}| \le \prod_{j=1}^{n} p^{j(r_p-\alpha_j)} = p^{\frac{n(n+1)}{2}r_p-(\alpha_1+2\alpha_2+\cdots+n\alpha_n)}. \tag{7.13}$$

Proof. By Proposition 5.1,

$$f_p(\begin{pmatrix} D & rSD^{-1} \\ O & rD^{-1}\end{pmatrix}) \neq 0 \iff \begin{pmatrix} D & rSD^{-1} \\ O & rD^{-1}\end{pmatrix} \in C_p.$$

Because $C_p \subseteq M_{2n}(\mathbf{Z}_p)$, we see that

$$|I_{A,p}| \le \text{meas}\{S \in S_n(\mathbf{Q}_p) |\, p^{r_p} SD^{-1} \in M_n(\mathbf{Z}_p)\}.$$

Writing $S = (s_{ij})$,

$$\begin{aligned} |I_{A,p}| &\le \text{meas}\{S \in S_n(\mathbf{Q}_p) |\, s_{ij}p^{r_p-\alpha_j} \in \mathbf{Z}_p \text{ for all } i, j\} \\ &= \text{meas}\{S \in S_n(\mathbf{Q}_p) |\, s_{ij} \in p^{-(r_p-\alpha_j)}\mathbf{Z}_p \text{ for all } 1 \le i \le j \le n\} \end{aligned}$$

since $s_{ij} = s_{ji}$. Hence,

$$|I_{A,p}| \le \prod_{j=1}^{n}\prod_{i\le j} \text{meas}(p^{-(r_p-\alpha_j)}\mathbf{Z}_p) = \prod_{j=1}^{n}\prod_{i\le j} p^{r_p-\alpha_j} = \prod_{j=1}^{n} p^{j(r_p-\alpha_j)}. \qquad \square$$

8. Weighted equidistribution of Satake parameters

Let $\widehat{\mathbb{G}} = \text{Spin}(2n+1, \mathbf{C})$ be the complex dual group of $\mathbb{G} = \text{PGSp}_{2n}$.† Since we are assuming trivial central character, the Satake parameters t_{π_p} belong to the maximal torus $\widehat{\mathbb{T}}$ of $\widehat{\mathbb{G}}$. For our fixed finite set $\mathbf{S}$ of primes, let

$$\mathfrak{X}_{\mathbf{S}} = (\widehat{\mathbb{T}}/W)^{|\mathbf{S}|}.$$

Each $\pi \in \Pi_{\mathbf{k}}(N)$ determines a point

$$t_\pi = (t_{\pi_p})_{p\in\mathbf{S}} \in \mathfrak{X}_{\mathbf{S}}. \tag{8.1}$$

By Shin's theorem, the points t_π become equidistributed relative to the Plancherel product measure on $\mathfrak{X}_{\mathbf{S}}$ as $N \to \infty$. Here we investigate their distribution with certain prescribed harmonic weights.

Proposition 8.1. *There exists a compact subset $\Omega \subseteq \mathfrak{X}_{\mathbf{S}}$ such that $t_\pi \in \Omega$ for all $\pi \in \Pi_{\mathbf{k}}(N)$.*

Proof. [BW, Theorem XI.3.3]. $\square$

†For dual groups, we always take the ground field to be $\mathbf{C}$ unless specified otherwise.

8.1. **Preliminary result.** For $\sigma_1, \sigma_2 \in \mathcal{R}_n^+$, define the weight

$$w_\pi(\sigma_1, \sigma_2) = \sum_{\varphi \in E_{\mathtt{k}}(\pi, N)} \frac{c_{\sigma_1}(\varphi)\overline{c_{\sigma_2}(\varphi)}}{\|\varphi\|^2}.$$

We will show in this section that the Satake parameters t_π, weighted by $w_\pi(\sigma, \sigma)$, have a uniform distribution relative to a certain Radon measure in the limit as $N \to \infty$.

The following is essentially a restatement of Theorem 1.1 (see §7.3).

Lemma 8.2. *Let* $c_{n\mathtt{k}\sigma_1} = \dfrac{(\det \sigma_1)^{\mathtt{k}-(n+1)/2}}{\pi^{n(n-1)/4}(4\pi)^{n(n+1)/2-n\mathtt{k}} \prod_{j=1}^{n} \Gamma(\mathtt{k} - \frac{n+j}{2})}$. *Then*

$$\lim_{N\to\infty} \frac{1}{\psi(N)} \sum_{\pi \in \Pi_{\mathtt{k}}(N)} w_\pi(\sigma_1, \sigma_2) = \delta_{\mathtt{k}}(\sigma_1, \sigma_2) c_{n\mathtt{k}\sigma_1},$$

where $\delta_{\mathtt{k}}(\sigma_1, \sigma_2)$ *is defined in* (1.3). *In particular, if* $\sigma_1 = \sigma_2 = \sigma \in \mathcal{R}_n^+$ *satisfies* $\delta_{\mathtt{k}}(\sigma, \sigma) > 0$ *(e.g. if* $\mathtt{k}$ *is even), then setting* $w_\pi = w_\pi(\sigma, \sigma)$ *we have*

$$0 < \lim_{N\to\infty} \frac{1}{\psi(N)} \sum_{\pi \in \Pi_{\mathtt{k}}(N)} w_\pi < \infty. \tag{8.2}$$

Remark: If $\mathtt{k}$ is even, then $\delta_{\mathtt{k}}(\sigma, \sigma) > 0$ since $A = I_n$ satisfies ${}^tA\sigma A = \sigma$. Hence in this case we always have nonvanishing in (8.2). This nonvanishing is crucial in what follows.

For the compact space Ω in Proposition 8.1, let

$$V_{\mathcal{S}} \subseteq C(\Omega) \tag{8.3}$$

denote the subspace consisting of all restrictions $F|_\Omega$ of functions $F = \prod_{p \in \mathbf{S}} \mathcal{S} f_p$ in the image $\prod_{p\in\mathbf{S}} \mathbf{C}[X^*(\widehat{\mathbb{T}})]^W$ of the $\mathbf{S}$-product Satake transform. (See (2.3) and (2.4).) For fixed $\mathtt{k} > 2n$ and $\sigma \in \mathcal{R}_n^+$ for which $\delta_{\mathtt{k}}(\sigma, \sigma) > 0$, let $w_\pi = w_\pi(\sigma, \sigma)$ as above. Then we may define a linear functional $\mathcal{L}$ on $V_{\mathcal{S}}$ by

$$\mathcal{L}(F) = \lim_{\substack{N\to\infty \\ (N,\mathbf{S})=1}} \frac{\sum_{\pi \in \Pi_{\mathtt{k}}(N)} w_\pi F(t_\pi)}{\sum_{\pi \in \Pi_{\mathtt{k}}(N)} w_\pi}. \tag{8.4}$$

By Lemma 8.2 and Theorem 6.3, the limit exists and is finite. Endowing $C(\Omega)$ with the L^∞ norm, it contains $V_{\mathcal{S}}$ as a dense subalgebra by the Stone-Weierstrass Theorem [Ru, p. 122]. (The latter algebra evidently separates points, and it is closed under complex conjugation by Proposition 2.1.) Because $V_{\mathcal{S}}$ is dense in $C(\Omega)$, the right-hand side of (8.4) exists for $F \in C(\Omega)$ (for details, see e.g. [KL1], pages 358-359). Moreover it is clear from (8.4) that $|\mathcal{L}(F)| \le \|F\|_\infty$ for all $F \in C(\Omega)$, so $\mathcal{L}$ is bounded. By the Riesz representation theorem, there exists a unique Radon measure $\mu = \mu_\sigma$ on Ω such that

$$\mathcal{L}(F) = \int_\Omega F\, d\mu \tag{8.5}$$

for all $F \in C(\Omega)$. This proves the following.

Theorem 8.3. *Let* $\mathtt{k} > 2n$, *and let* σ *be a symmetric positive-definite half-integral matrix for which* $\delta_{\mathtt{k}}(\sigma, \sigma) > 0$. *(This is automatic, for example, if* $\mathtt{k}$ *is even.) Then the Satake parameters* $(t_\pi)_{\pi \in \Pi_{\mathtt{k}}(N)}$ *of* (8.1), *when weighted by* $w_\pi(\sigma, \sigma)$, *become equidistributed with respect to the above measure* μ_σ *in the limit as* $N \to \infty$ *along integers coprime to* $\mathbf{S}$.

Of course, one would like to know more about the measure μ, for example whether it is supported on the tempered spectrum. (Recall that an unramified representation π_p of $G(\mathbf{Q}_p)$ is tempered if and only if its Satake parameter t_{π_p} lies in a compact subgroup of $\widehat{\mathbb{T}}$.) We will pursue this question by relating μ to the Sato-Tate measure, which is supported in a compact subtorus of $\widehat{\mathbb{T}}$. See Theorem 8.4 below.

8.2. **Relating two measures.** Generally, suppose η is a Radon measure on Ω, and $\{R_\lambda\}_{\lambda\in\Lambda}$ is a set of continuous functions forming an orthonormal basis for $L^2(\Omega,\eta)$ which also spans an L^∞-dense subspace of $C(\Omega)$. Then the measure μ in (8.5) can be expressed as

$$d\mu(t) = \sum_\lambda \mathcal{L}(R_\lambda)\overline{R_\lambda}(t)d\eta(t), \tag{8.6}$$

provided the sum is uniformly absolutely convergent on Ω. Indeed, for all $\alpha \in \Lambda$,

$$\int_\Omega R_\alpha(t)\sum_\lambda \mathcal{L}(R_\lambda)\overline{R_\lambda}(t)d\eta = \sum_\lambda \mathcal{L}(R_\lambda)\left\langle R_\alpha, R_\lambda\right\rangle_\eta = \mathcal{L}(R_\alpha) = \int_\Omega R_\alpha d\mu,$$

and by linearity and density of the span of the R_α, the above holds as well for all functions in $C(\Omega)$.

8.3. **The Sato-Tate measure.** Fix a maximal compact subgroup $\widehat{H} \subseteq \widehat{\mathbb{G}}$ with maximal torus

$$\widehat{\mathbb{T}}_c = \widehat{H} \cap \widehat{\mathbb{T}} = \widehat{\mathbb{T}}(\mathbf{C})^1.$$

This is the maximal compact subtorus of $\widehat{\mathbb{T}}$. Let dh denote the Haar measure on $\widehat{H}$ of total volume 1. Because every conjugacy class in $\widehat{H}$ contains exactly one Weyl orbit of $\widehat{\mathbb{T}}_c$, the measure dh induces a quotient measure μ_{ST} on the space $\widehat{\mathbb{T}}_c/W$. We extend μ_{ST} to $\widehat{\mathbb{T}}/W$ by taking it to be zero on the complement of $\widehat{\mathbb{T}}_c/W$. This is the Sato-Tate measure. In more detail, for $f \in C(\widehat{\mathbb{T}}_c/W)$, we may identify f with a class function on $\widehat{H}$, and

$$\int_{\widehat{\mathbb{T}}_c/W} f(t)d\mu_{ST}(t) = \int_{\widehat{H}} f(h)dh. \tag{8.7}$$

By the Weyl integration formula, the measure is given explicitly by

$$d\mu_{ST}(t) = \Big|\det(\mathrm{Ad}(t^{-1}-I)|_{\mathrm{Lie}(\widehat{H})/\,\mathrm{Lie}(\widehat{\mathbb{T}}_c)})\Big|dt, \tag{8.8}$$

where dt is the Haar measure giving $\widehat{\mathbb{T}}_c$ volume 1. An alternative expression for it is given in (8.13) below.

Fix a set Φ^+ of positive roots in the root system attached to $\widehat{H}$ and $\widehat{\mathbb{T}}_c$. We shall identify $X^*(\widehat{\mathbb{T}})$ and $X^*(\widehat{\mathbb{T}}_c)$. By the theorem of the highest weight, the irreducible representations π_λ of $\widehat{H}$ are in one-to-one correspondence with the elements $\lambda \in \mathcal{C}^+$, where $\mathcal{C}^+$ is the positive Weyl chamber of $X^*(\widehat{\mathbb{T}}_c) = X^*(\widehat{\mathbb{T}}) \cong X_*(\mathbb{T})$ given in (3.18). Let

$$F_\lambda = \operatorname{tr}\pi_\lambda \tag{8.9}$$

denote the trace of π_λ. It is a class function on $\widehat{H}$, so we may view it as a function on $\widehat{\mathbb{T}}_c/W$. By the Peter-Weyl theorem, the set $\{F_\lambda|\, \lambda \in \mathcal{C}^+\}$ is an orthonormal basis for the space of L^2 class functions on $\widehat{H}$ (relative to the measure dh). In particular, by (8.7) we have

$$\int_{\widehat{\mathbb{T}}(\mathbf{C})/W} F_\lambda(t)\overline{F_\mu(t)}\, d\mu_{ST} = \delta_{\lambda,\mu} \tag{8.10}$$

(for the Kronecker δ). Here, the domain of F_λ is extended from $\widehat{\mathbb{T}}_c$ to $\widehat{\mathbb{T}}(\mathbf{C})$ by viewing F_λ as a sum

$$F_\lambda = \sum_{\mu \in X^*(\widehat{\mathbb{T}})} m_\lambda(\mu)[\mu] \in \mathbf{C}[X^*(\widehat{\mathbb{T}})]^W.$$

The orthogonality (8.10) can also be proved using (8.12) and (8.13) below.

We shall need the fact that the set $\{F_\lambda | \lambda \in \mathcal{C}^+\}$ spans $\mathbf{C}[X^*(\widehat{\mathbb{T}})]^W$ (see [FH, Theorem 23.24], using the fact that $\Lambda = X^*(\widehat{\mathbb{T}})$ since $\widehat{\mathbb{G}} = \mathrm{Spin}(2n+1)$ is simply connected). By (2.4), this space coincides with the image of the local Satake transform.

Given a tuple $\underline{\lambda} = (\lambda_p) \in \prod_{p \in \mathbf{S}} \mathcal{C}^+$, we let

$$F_{\underline{\lambda}} = \prod_{p \in \mathbf{S}} F_{\lambda_p} \in \prod_{p \in \mathbf{S}} \mathbf{C}[X^*(\widehat{\mathbb{T}})]^W.$$

Viewing the $F_{\underline{\lambda}}$ as functions on Ω (by restriction), they span the space $V_{\mathcal{S}}$ of (8.3). This follows from the above discussion.

8.4. **Relation between μ and μ_{ST}.** We continue to assume that $\delta_{\mathbf{k}}(\sigma, \sigma) > 0$, so that for $F_{\underline{\lambda}}$ as above, we may consider $\mathcal{L}(F_{\underline{\lambda}})$ as in (8.4).

Theorem 8.4. *Let $\rho \in X^*(\mathbb{T}) = X_*(\widehat{\mathbb{T}})$ be half the sum of the positive roots, as in* (3.9). *Suppose that there exits $\varepsilon > 0$ such that for all tuples $\underline{\lambda}$ as above,*

$$\mathcal{L}\Big(\prod_{p \in \mathbf{S}} \mathcal{S}(c_{\lambda_p})\Big) \ll_\varepsilon \prod_{p \in \mathbf{S}} p^{(1-\varepsilon)\langle \rho, \lambda_p \rangle}, \tag{8.11}$$

where c_{λ_p} is the characteristic function of $K_p \lambda_p(p) K_p$. Then the measure μ defined in (8.5) *is given by*

$$d\mu(t) = \sum_{\underline{\lambda}} \mathcal{L}(F_{\underline{\lambda}}) \overline{F_{\underline{\lambda}}(t)}\, d\mu_{\mathbf{S}}(t),$$

where $\mu_{\mathbf{S}} = \prod_{p \in \mathbf{S}} \mu_{ST}$ is the product measure on $\mathfrak{X}_{\mathbf{S}}$, and the above sum converges absolutely and uniformly on Ω.

Remarks: (1) Hypothesis (8.11) would follow from (a) adequate bounds on the number of matrices A satisfying the conditions of Theorem 6.3, and (b) adequate bounds for the local geometric integrals $I_{A,p}(c_{\lambda_p})$ for $p \in \mathbf{S}$. (See (9.3).) In Section 9, we will carry this out and prove Hypothesis (8.11) in the special case $n = 2$, as an application of Theorem 6.3.

(2) It is not clear to us whether there is a closed form expression for the measure.

Proof. In Section 8.3, we saw that the set $\{F_\lambda | \lambda \in \mathcal{C}^+\}$ is an orthonormal basis for $L^2(\Omega, \mu_{ST})$. Furthermore, it is dense in $V_{\mathcal{S}}$, which in turn is dense in $C(\Omega)$ as discussed before Theorem 8.3. Hence, by the discussion in §8.2, it suffices to prove that the given series is uniformly convergent under Hypothesis (8.11).

To ease the notation, we will first assume that $\mathbf{S} = \{p\}$ consists of just one prime. For any weight $\lambda \in X^*(\widehat{\mathbb{T}})$, define

$$A_\lambda = \sum_{w \in W} (\operatorname{sgn} w) w(\lambda) \in \mathbf{C}[X^*(\widehat{\mathbb{T}})].$$

For $t \in \widehat{\mathbb{T}}_c$,

$$|A_\lambda(t)| \le \left| \sum_{w \in W} (\operatorname{sgn} w) w(\lambda)(t) \right| \le \sum_{w \in W} |w(\lambda)(t)| = |W|.$$

By the Weyl character formula ([FH, Theorem 24.2]),

$$F_\lambda = \frac{A_{\lambda+\rho}}{A_\rho}. \tag{8.12}$$

It is well-known that

$$d\mu_{ST}(t) = |A_\rho(t)|^2 dt. \tag{8.13}$$

(For example, compare (25.6) of [Bu] ((22.7) in the 2nd edition) with Lemma 24.3 of [FH]).

Therefore, for $t \in \widehat{\mathbb{T}}_c$, we need to prove the convergence of

$$\sum_{\lambda\in\mathcal{C}^+} |\mathcal{L}(F_\lambda)\overline{F_\lambda(t)}||A_\rho(t)|^2 = \sum_{\lambda\in\mathcal{C}^+} |\mathcal{L}(F_\lambda)\overline{A_{\lambda+\rho}(t)}A_\rho(t)| \le |W|^2 \sum_{\lambda\in\mathcal{C}^+} |\mathcal{L}(F_\lambda)|.$$

Next, we need to relate F_λ to the functions $\mathcal{S}(c_\mu)$ in order to make use of Hypothesis (8.11). This is achieved by the following formula of Kato and Lusztig, which holds in any split reductive p-adic group ([HKP, Theorem 7.8.1]; see also [Gr, (3.12) and Proposition 4.4]):

$$F_\lambda = p^{-\langle\lambda,\rho\rangle} \sum_{\mu\le\lambda} P_{\mu,\lambda}(p)\mathcal{S}(c_\mu).$$

Here, μ belongs to $\mathcal{C}^+$, and $P_{\mu,\lambda}$ is the Kazhdan-Lusztig polynomial

$$P_{\mu,\lambda}(p) = p^{\langle\lambda-\mu,\rho\rangle} \sum_{w\in W} \mathrm{sgn}(w)\widehat{P}(w(\lambda+\rho^\vee) - (\mu+\rho^\vee)),$$

where

$$\widehat{P}(\mu) = \sum_{\mu=\sum n(\alpha^\vee)\alpha^\vee} p^{-\sum n(\alpha^\vee)} \ge 0$$

encodes the number of expressions of μ as a linear combination of positive co-roots with coefficients $n(\alpha^\vee) \ge 0$. We note that $P_{\lambda,\lambda}(p) = 1$, [Gr, (4.5)].

Therefore, the quantity we need to bound is

$$\sum_{\lambda\in\mathcal{C}^+} |\mathcal{L}(F_\lambda)| = \sum_{\lambda\in\mathcal{C}^+} \left| p^{-\langle\lambda,\rho\rangle} \sum_{\mu\le\lambda} P_{\mu,\lambda}(p)\mathcal{L}(\mathcal{S}(c_\mu))\right|$$

$$\le \sum_{\mu\in\mathcal{C}^+} p^{-\langle\mu,\rho\rangle} |\mathcal{L}(\mathcal{S}(c_\mu))| \sum_{\lambda\ge\mu}\sum_{w\in W} \widehat{P}(w(\lambda+\rho^\vee) - (\mu+\rho^\vee)). \tag{8.14}$$

We claim that for $w \in W$,

$$\sum_{\lambda\ge\mu} \widehat{P}(w(\lambda+\rho^\vee) - (\mu+\rho^\vee)) \le 2^{d^+}, \tag{8.15}$$

where d^+ is the number of positive co-roots. Indeed, the left-hand side of (8.15) is

$$\sum_{\lambda\ge\mu} \sum_{\sum n(\alpha^\vee)\alpha^\vee} p^{-\sum n(\alpha^\vee)}, \tag{8.16}$$

where the inner sum is extended over all expressions of the form

$$w(\lambda+\rho^\vee) - (\mu+\rho^\vee) = \sum_{\alpha^\vee} n(\alpha^\vee)\alpha^\vee$$

with $n(\alpha^\vee) \ge 0$ and $\alpha^\vee$ positive co-roots. The above expression is equivalent to

$$\lambda = w^{-1}(\sum_{\alpha^\vee} n(\alpha^\vee)\alpha^\vee + (\mu+\rho^\vee)) - \rho^\vee. \tag{8.17}$$

Thus we may exchange the order of summation in (8.16), so the left-hand side of (8.15) is equal to

$$\sum\nolimits^{*}_{\sum n(\alpha^\vee)\alpha^\vee} p^{-\sum n(\alpha^\vee)},$$

where the $*$ indicates that we consider only those expressions for which the right-hand side of (8.17) is $\geq \mu$. The above is of course bounded by the sum over *all* nonnegative linear combinations of positive co-roots

$$\sum_{\sum n(\alpha^\vee)\alpha^\vee} p^{-\sum n(\alpha^\vee)} = \prod_{\alpha^\vee} \sum_{n(\alpha^\vee)=0}^{\infty} p^{-n(\alpha^\vee)} \leq \prod_{\alpha^\vee} 2,$$

proving the claim (8.15).

Combining (8.14) and (8.15), it follows that

$$\sum_{\lambda\in\mathcal{C}^+} |\mathcal{L}(F_\lambda)| \ll |W| \sum_{\mu\in\mathcal{C}^+} p^{-\langle\mu,\rho\rangle} |\mathcal{L}(\mathcal{S}(c_\mu))|.$$

Using the given bound (8.11), the above is

$$\ll \sum_{\mu\in\mathcal{C}^+} p^{-\varepsilon\langle\mu,\rho\rangle}.$$

There exists a finite set $\{\mu_1,\ldots,\mu_\ell\} \subseteq \mathcal{C}^+$ such that $\mathcal{C}^+ \subseteq \{\sum_{i=1}^{\ell} a_i\mu_i \,|\, 0 \leq a_i \in \mathbf{Z}\}$. Writing $\mu = \sum_{i=1}^{\ell} a_i\mu_i$, the above is

$$\leq \prod_{i=1}^{\ell} \left(\sum_{a_i=0}^{\infty} p^{-\varepsilon\langle\mu_i,\rho\rangle a_i} \right) < \infty.$$

This completes the proof when $\mathbf{S} = \{p\}$.

The general case is proven in the same way, using

$$\begin{aligned} \sum_{\underline{\lambda}} |\mathcal{L}(F_{\underline{\lambda}})| &= \sum_{\underline{\lambda}} \left| \sum_{\underline{\mu}\leq\underline{\lambda}} \left(\prod_{p\in\mathbf{S}} p^{-\langle\lambda_p,\rho\rangle} P_{\mu_p,\lambda_p}(p) \right) \mathcal{L}(\prod_{p\in\mathbf{S}} \mathcal{S}(c_{\mu_p})) \right| \\ &\leq \sum_{\underline{\mu}} |\mathcal{L}(\prod_{p\in\mathbf{S}} \mathcal{S}(c_{\mu_p}))| \prod_{p\in\mathbf{S}} \sum_{\lambda_p\geq\mu_p} p^{-\langle\lambda_p,\rho\rangle} |P_{\mu_p,\lambda_p}(p)| \\ &\ll \sum_{\underline{\mu}} |\mathcal{L}(\prod_{p\in\mathbf{S}} \mathcal{S}(c_{\mu_p}))| \prod_{p\in\mathbf{S}} p^{-\langle\mu_p,\rho\rangle}. \end{aligned}$$

Using Hypothesis 8.11, one shows as before that this is finite. □

Corollary 8.5. *Write $\mu = \mu_p$ for the measure on $\widehat{\mathbb{T}}/W$ defined in* (8.5) *when $\mathbf{S} = \{p\}$. Then under Hypothesis 8.11,*

$$\lim_{p\to\infty} d\mu_p(t) = d\mu_{ST}(t).$$

Proof. Let $\mathbf{0} \in \mathcal{C}^+$ denote the element corresponding to the zero vector in $\mathbf{Z}^{n+1}$. As in the proof of the previous proposition,

$$\sum_{\lambda\in\mathcal{C}^+-\{\mathbf{0}\}} |\mathcal{L}(F_\lambda)\overline{F_\lambda(t)}||A_\rho(t)|^2 \ll \sum_{\mu\in\mathcal{C}^+-\{\mathbf{0}\}} p^{-\varepsilon\langle\mu,\rho\rangle}.$$

Noting that $\langle \mu, \rho \rangle > 0$ when $\mu \neq \mathbf{0}$, the right-hand side tends to 0 as p goes to ∞. Thus

$$\lim_{p\to\infty} d\mu_p(t) = \lim_{p\to\infty} \sum_{\lambda\in\mathcal{C}^+} \mathcal{L}(F_\lambda)\overline{F_\lambda(t)}\, d\mu_{ST}(t) = \mathcal{L}(F_{\mathbf{0}})F_{\mathbf{0}}(t)d\mu_{ST}(t) = d\mu_{ST}(t).$$

The last step follows by (8.4) and the fact that $F_{\mathbf{0}} = 1$ (cf. (8.12)). □

9. Local computation when $n = 2$

Here we refine the discussion from Section 7.4 for $p \in \mathbf{S}$, with the simplifying assumptions that $n = 2$, and

$$p \nmid 4 \det \sigma_1. \tag{9.1}$$

(Recall that $\det \sigma_1 \in \frac{1}{4}\mathbf{Z}$.) The main goal of this section is to prove the following local bound.

Proposition 9.1. *Under the above hypotheses, there exists a constant $\varepsilon > 0$ such that*

$$|I_{A,p}(c_\lambda)| \ll p^{(1-\varepsilon)\langle\lambda,\rho\rangle - \varepsilon r_p} \tag{9.2}$$

for all $\lambda \in \mathcal{C}^+$, where the implied constant depends only on p and ε.

Before proving the proposition, let us observe how it implies the global Hypothesis (8.11).

Corollary 9.2. *Suppose $n = 2$, $\sigma_1 = \sigma_2 = \sigma$, $\delta_{\mathbf{k}}(\sigma,\sigma) > 0$, and $p \nmid 4\det\sigma$ for all $p \in \mathbf{S}$. Then Hypothesis* (8.11) *holds.*

Proof. In Section 7.2, we saw that $I_{A,p} = 1$ for primes $p \notin \mathbf{S}$. From the definition (8.4) of $\mathcal{L}$, Theorem 6.3, and Lemma 8.2, it follows that

$$\mathcal{L}\Big(\prod_{p\in\mathbf{S}} \mathcal{S}(c_{\lambda_p})\Big) = \frac{\sum_A c_{n\mathbf{k}\sigma} \prod_{p\in\mathbf{S}} I_{A,p}(c_{\lambda_p})}{\delta_{\mathbf{k}}(\sigma,\sigma) c_{n\mathbf{k}\sigma}} = \frac{1}{\delta_{\mathbf{k}}(\sigma,\sigma)} \sum_A \prod_{p\in\mathbf{S}} I_{A,p}(c_{\lambda_p}), \tag{9.3}$$

where A runs through the rank-2 matrices in $M_2(\mathbf{Z})/\{\pm 1\}$ satisfying $r\,{}^tA^{-1} \in M_2(\mathbf{Z})$ and ${}^tA\sigma A = r\sigma$. In particular, writing

$$\sigma = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \qquad A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_2(\mathbf{Z}),$$

we have $ax^2 + bxz + cz^2 = ra$. Hence

$$4ra^2 = (2ax+bz)^2 - (b^2-4ac)z^2 = (2ax + bz + \sqrt{D}z)(2ax + bz - \sqrt{D}z),$$

where $D = b^2 - 4ac < 0$. Thus, in the ring of integers $\mathcal{O} \subseteq \mathbf{Q}[\sqrt{D}]$, the ideal $(2ax+bz+\sqrt{D}z)$ is a factor of the ideal $(4ra^2)$. The number of ideal factors of $(4ra^2)$ is $\ll r^{\varepsilon/2}$. In view of the fact that $|\mathcal{O}^*| < \infty$, the number of possible choices for x, z is $\ll r^{\varepsilon/2}$. Similarly, the number of choices for y, w is $\ll r^{\varepsilon/2}$. So the number of terms in the sum is $\ll r^\varepsilon = \prod_{p\in\mathbf{S}} p^{r_p\varepsilon}$. It follows from (9.2) that the above is

$$\ll \prod_{p\in\mathbf{S}} p^{(1-\varepsilon)\langle\lambda_p,\rho\rangle},$$

as required. □

The proof of Proposition 9.1 is given in Section 9.3. In the intervening sections, we describe how to compute the local integral $I_{A,p}$ explicitly, with the goal of producing the upper bound (9.2). In many situations, the trivial bound (7.13) is adequate, so an explicit computation is not necessary. In the remaining cases (which, in the notation below, occur when $\beta - 1 \leq t$), we give a complete treatment of the local integral.

9.1. **Preliminaries.** Without loss of generality, we consider the case where $f_p = c_\lambda$ is the characteristic function of the double coset $Z_p K_p \lambda(p) K_p$, where

$$\lambda(p) = \operatorname{diag}(1, p^t, p^\tau, p^{\tau-t}) \tag{9.4}$$

for $0 \le t \le \tau/2$ as in (3.20). Thus we write τ in place of r_p (for the purpose of eliminating a subscript).

By Proposition 7.4, we need only consider matrices D of the form

$$D = \operatorname{diag}(p^\alpha, p^\beta), \quad 0 \le \alpha \le \beta, \quad \alpha + \beta = \tau.$$

Write

$$\sigma_U = {}^tU \sigma_1 U = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

Note that σ_U is half-integral, and $\det \sigma_U = \det \sigma_1$. So by (9.1), either $p \nmid b$ or $p \nmid ac$. We would like to compute the integral

$$I_{A,p} = \int_{S(\mathbf{Q}_p)} f_p(\begin{pmatrix} D & p^\tau S D^{-1} \\ O & p^\tau D^{-1} \end{pmatrix}) \theta_p(\operatorname{tr} \sigma_U S) dS. \tag{9.5}$$

Writing $S = \begin{pmatrix} x & y \\ y & z \end{pmatrix}$, we let

$$M = \begin{pmatrix} D & p^\tau S D^{-1} \\ O & p^\tau D^{-1} \end{pmatrix} = \begin{pmatrix} p^\alpha & 0 & p^\beta x & p^\alpha y \\ 0 & p^\beta & p^\beta y & p^\alpha z \\ 0 & 0 & p^\beta & 0 \\ 0 & 0 & 0 & p^\alpha \end{pmatrix}.$$

By Proposition 3.2, $M \in \operatorname{supp} f_p$ if and only if the fractional ideal generated by all the entries is $(1) = \mathbf{Z}_p$ and the fractional ideal generated by the determinants of all 2×2 submatrices is $(p^t) = p^t \mathbf{Z}_p$. The determinants of the 2×2 submatrices of M are shown in the table below:

rows \ cols	1,2	1,3	1,4	2,3	2,4	3,4
1,2	$p^{\alpha+\beta}$	$p^{\alpha+\beta} y$	$p^{2\alpha} z$	$-p^{2\beta} x$	$-p^{\alpha+\beta} y$	$p^{\alpha+\beta}(xz - y^2)$
1,3	0	$p^{\alpha+\beta}$	0	0	0	$-p^{\alpha+\beta} y$
1,4	0	0	$p^{2\alpha}$	0	0	$p^{\alpha+\beta} x$
2,3	0	0	0	$p^{2\beta}$	0	$-p^{\alpha+\beta} z$
2,4	0	0	0	0	$p^{\alpha+\beta}$	$p^{\alpha+\beta} y$
3,4	0	0	0	0	0	$p^{\alpha+\beta}$

Using $\alpha \le \beta$ and $\alpha + \beta = \tau$, we see that $M \in \operatorname{supp} f_p$ if and only if

$$(p^\alpha,\, p^\beta x,\, p^\alpha y,\, p^\alpha z) = (1) \tag{9.6}$$

and

$$(p^{2\alpha},\, p^\tau x,\, p^\tau y,\, p^{2\alpha} z,\, p^\tau (xz - y^2)) = (p^t). \tag{9.7}$$

Let

$$x' = p^\beta x, \quad y' = p^\alpha y, \quad z' = p^\alpha z. \tag{9.8}$$

Then (9.6) is equivalent to

$$(p^\alpha, x', y', z') = (1) \tag{9.9}$$

and (9.7) is equivalent to

$$(p^{2\alpha}, p^\alpha x', p^\beta y', p^\alpha z', x'z' - p^{\beta-\alpha}{y'}^2) = (p^t). \tag{9.10}$$

If $\alpha \neq 0$, then (9.6) is equivalent to

$$(x', y', z') = (1), \tag{9.11}$$

i.e., $x', y', z' \in \mathbf{Z}_p$ and at least one of them is a unit.

9.2. **Evaluation of the integral $I_{A,p}$.** We continue with the notation from above. Given a Borel subset $S' \subseteq S(\mathbf{Q}_p)$, we define

$$I_{S'} = \int_{S'} f_p(\begin{pmatrix} D & rSD^{-1} \\ O & rD^{-1} \end{pmatrix})\theta_p(\mathrm{tr}\, \sigma_U S) dS.$$

Define

$$S_0 = \{\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S(\mathbf{Q}_p) |\, x, y, z \text{ satisfy (9.6) and (9.7)}\}.$$

Then $I_{S_0} = I_{A,p}$ is the integral (9.5) we need to compute.

Let $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $E'_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Proposition 9.3. *Let S' be a Borel subset of S_0. Suppose $p \nmid a$ (resp. $p \nmid b$, $p \nmid c$). Suppose $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S'$ implies that $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E_{11}$ ($\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E'_{12}$, $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E_{22}$ respectively) belongs to S'. Then $I_{S'} = 0$.*

Proof. Suppose $p \nmid a$. The other cases can be handled similarly. By the given property, $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S'$ if and only if $\begin{pmatrix} x & y \\ y & z \end{pmatrix} + \frac{1}{p}E_{11} \in S'$. Hence

$$I_{S'} = \int_{S'} \theta_p(\mathrm{tr}\, \sigma_U S) dS = \int_{S'} \theta_p(\mathrm{tr}\, \sigma_U (S - \frac{1}{p}E_{11})) dS = e(\frac{a}{p}) I_{S'}.$$

The proposition follows. □

Proposition 9.4. *Suppose $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S_0$. Then:*

(i) *If $\beta \geq 2$, $\tau - 1 \geq t + 1$ and $p^{t+1} | p^{\beta-1}z'$, then $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E_{11} \in S_0$.*

(ii) *If $\alpha \geq 2$, $\tau - 2 \geq t + 1$ and $p^{t+1} | p^{\beta-1}y'$, then $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E'_{12} \in S_0$.*

(iii) *If $\alpha \geq 2$, $2\alpha - 1 \geq t + 1$, $p^{t+1} | p^{\alpha-1}x'$, then $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E_{22} \in S_0$.*

Remarks: 1) If in (9.10), $(p^{2\alpha}, p^\alpha x', p^\beta y', p^\alpha z') = (p^t)$, then the last condition of (i) (resp. (ii), (iii)) can be replaced by the weaker condition $p^t | p^{\beta-1}z'$ (resp. $p^t | p^{\beta-1}y'$, $p^t | p^{\alpha-1}x'$), and the second condition of (ii) can be weakened to $\tau - 2 \geq t$.

2) There are some other variants; for example, if $(x', z') = 1$ and $\alpha = t$, then in (ii), the conditions can be replaced by $\alpha \geq 1$, $\tau - 2 \geq t$, and $p^t | p^{\beta-1}y'$.

Proof. (i) Replace x by $x \pm \frac{1}{p}$ in (9.6) and (9.7). The left-hand side of (9.6) becomes

$$(p^\alpha, p^\beta x \pm p^{\beta-1}, p^\alpha y, p^\alpha z).$$

The left-hand side of (9.7) becomes

$$(p^{2\alpha}, p^\tau x \pm p^{\tau-1}, p^\tau y, p^{2\alpha} z, p^\tau(xz-y^2) \pm p^{\beta-1}z').$$

Under the given hypotheses, $p^{\beta-1} \in (p)$, $p^{\tau-1} \in (p^{t+1})$ and $p^{\beta-1}z' \in (p^{t+1})$. Hence $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E_{11}$ satisfies (9.6) and (9.7).

(ii) Replace y by $y \pm \frac{1}{p}$ in (9.6) and (9.7). The left-hand side of (9.6) becomes

$$(p^\alpha, p^\beta x, p^\alpha y \pm p^{\alpha-1}, p^\alpha z).$$

The left-hand side of (9.7) becomes

$$(p^{2\alpha}, p^\tau x, p^\tau y \pm p^{\tau-1}, p^{2\alpha} z, p^\tau(xz-y^2) \mp 2p^{\beta-1}y' - p^{\tau-2}).$$

Under the given hypotheses in this case, $p^{\alpha-1} \in (p)$, $p^{\tau-2} \in (p^{t+1})$ and $p^{\beta-1}y' \in (p^{t+1})$. Hence $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E'_{12}$ satisfies (9.6) and (9.7).

Assertion (iii) and the remarks can be proven similarly. □

Corollary 9.5. *Suppose $p \nmid a$, and that*

(i) $\beta \geq 2$,
(ii) $\tau - 1 \geq t + 1$,
(iii) $\beta - 1 \geq t + 1$.

Then $I_{A,p} = 0$.

Proof. Suppose $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S_0$. Then by (9.9), $z' \in \mathbf{Z}_p$, so by the third hypothesis, $p^{t+1} | p^{\beta-1}z'$. By Proposition 9.4, $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E_{11} \in S_0$. The assertion now follows by Proposition 9.3. □

Corollary 9.6. *Suppose $p \nmid b$, and*

(i) $\alpha \geq 2$,
(ii) $\tau - 2 \geq t + 1$,
(iii) $\beta - 1 \geq t + 1$.

Then $I_{A,p} = 0$.

Proof. This follows in the same way as the previous corollary, using Proposition 9.4 (ii), and Proposition 9.3. □

Proposition 9.7. *Suppose condition (iii) of the above corollaries fails to hold, i.e., $\beta - 1 \leq t$. Then exactly one of the following is true:*

(1) $\tau = 2\tau' + 1$ *is odd,* $\alpha = \tau'$, $\beta = \tau' + 1$, *and* $t = \tau'$,
(2) $\tau = 2\tau'$ *is even,* $\alpha = \tau' - 1$, $\beta = \tau' + 1$, *and* $t = \tau'$,
(3) $\tau = 2\tau'$ *is even, and* $\alpha = \beta = t = \tau'$,
(4) $\tau = 2\tau'$ *is even,* $\alpha = \beta = \tau'$, *and* $t = \tau' - 1$.

Proof. Suppose $\beta \leq t + 1$. Then because we always have $t \leq \left[\frac{\tau}{2}\right]$ (where brackets denote the floor), it follows that $\beta \leq t + 1 \leq \left[\frac{\tau}{2}\right] + 1$. On the other hand, $\tau = \alpha + \beta \leq 2\beta$, which gives the lower bound in the following inequality:

$$\left\lceil \frac{\tau}{2} \right\rceil \leq \beta \leq \left[\frac{\tau}{2}\right] + 1.$$

(Here, $\lceil\cdot\rceil$ denotes the ceiling.) Using $\beta-1 \le t \le \left\lceil\frac{\tau}{2}\right\rceil$, the result follows easily by considering the possible cases. ☐

Proposition 9.8. *Suppose α,β,t,τ satisfy Proposition 9.7 (1), i.e., $\tau = 2\tau'+1$, $\alpha=\tau'$, $\beta=\tau'+1$, and $t=\tau'$. Then if $\tau'\ge 2$, $I_{A,p}=0$.*

Proof. Let $\begin{pmatrix} x & y \\ y & z\end{pmatrix} \in S_0$. Then (9.10) is satisfied, and since $\alpha=\tau'\ge 2$, (9.11) is also satisfied. In particular, by (9.10),

$$x'z' \equiv p{y'}^2 \pmod{p^2}. \tag{9.12}$$

It follows that either x' or z' is a unit. Indeed, if $p|x'$ and $p|z'$, then y' is a unit by (9.11), leading to an obvious contradiction in (9.12). In fact, by (9.12), $p|x'z'$ and hence exactly one of x' or z' is a unit.

First suppose $p\nmid b$. Note that $(x',z')=1$, $\alpha=t\ge 2$,

$$\tau-2 = 2\tau'-1 > 2\tau'-\tau' = t,$$

and $p^t|p^{\beta-1}y'$. By the second remark after Proposition 9.4, $\begin{pmatrix} x & y \\ y & z\end{pmatrix} \pm \frac{1}{p}E'_{12} \in S_0$. By Proposition 9.3, $I_{A,p}=I_{S_0}=0$.

Finally, suppose $p|b$. Then as noted earlier, $p\nmid a$. Since one of x' or z' is a unit, $(p^\alpha x', p^\alpha z') = (p^t)$. Furthermore, $p^t|p^{\beta-1}z'$, and as above, $\tau-2\ge t$. By the first remark after Proposition 9.4, $\begin{pmatrix} x & y \\ y & z\end{pmatrix} \pm \frac{1}{p}E_{11} \in S_0$. By Proposition 9.3, $I_{A,p}=I_{S_0}=0$. ☐

Proposition 9.9. *Suppose α,β,t,τ satisfy Proposition 9.7 (2), i.e., $\tau=2\tau'$ is even, $\alpha=\tau'-1$, $\beta=\tau'+1$, and $t=\tau'$. Then if $\tau'\ge 3$, $I_{A,p}=0$. In fact, if $p\nmid ac$, then $I_{A,p}=0$ if $\tau'\ge 2$.*

Proof. Let $\begin{pmatrix} x & y \\ y & z\end{pmatrix} \in S_0$. Suppose $\tau'\ge 2$. By (9.10), $p^{\tau'}|p^{\tau'-1}x'$ and $p^{\tau'}|p^{\tau'-1}z'$, and hence $p|x'$ and $p|z'$. Therefore by (9.11), y' is a unit.

Suppose $p\nmid ac$. Because $p|z'$, we have $p^{t+1}|p^{\beta-1}z'$. Hence by Proposition 9.4, $\left(\begin{smallmatrix} x & y \\ y & z\end{smallmatrix}\right) \pm \frac{1}{p}E_{11}\in S_0$. By Proposition 9.3, $I_{S_0}=0$.

Next suppose $p\nmid b$ and $\tau'\ge 3$. Write $x'=px''$ and $z'=pz''$, with x'' and $z''\in \mathbf{Z}_p$. By (9.10), $p^2x''z''\equiv p^2y'^2 \pmod{p^t}$, so $x''z''\equiv y'^2 \pmod{p^{\tau'-2}}$. Because $\tau'\ge 3$, it follows that x'' and z'' are units. Therefore $(p^\alpha x')=(p^t)$. Obviously $p^t|p^{\beta-1}y'$. By the first remark after Proposition 9.4, $\left(\begin{smallmatrix} x & y \\ y & z\end{smallmatrix}\right) \pm \frac{1}{p}E'_{12}\in S_0$. By Proposition 9.3, $I_{A,p}=I_{S_0}=0$. ☐

Proposition 9.10. *Suppose α,β,t,τ satisfy Proposition 9.7 (3), i.e. $\tau=2\tau'$ and $\alpha=\beta=t=\tau'$. Suppose further that $\tau'\ge 2$. Then the integral $I_{A,p}$ is given explicitly by (9.14) below.*

Proof. Suppose $\begin{pmatrix} x & y \\ y & z\end{pmatrix} \in S_0$. Then (9.11) implies $(p^tx',p^ty',p^tz')=(p^t)$. Hence (9.11) and (9.10) taken together are equivalent to (9.11) and

$$x'z' \equiv {y'}^2 \pmod{p^t}. \tag{9.13}$$

If y' is not a unit, then by (9.13) and (9.11), exactly one of x' or z' is a unit. So there is a partition

$$S_0 = S_1\cup S_2\cup S_3,$$

where

$$S_1 = \{\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S_0 : x', y', z' \text{ are units}\},$$
$$S_2 = \{\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S_0 : p|y',\ p|z', \text{ and } x' \text{ is a unit}\},$$
$$S_3 = \{\begin{pmatrix} x & y \\ y & z \end{pmatrix} \in S_0 : p|y',\ p|x', \text{ and } z' \text{ is a unit}\}.$$

We claim that $I_{S_2} = I_{S_3} = 0$. Let $\left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right) \in S_2$. Then $(p^\alpha x') = (p^t)$, $p^t | p^{\beta-1} z'$, and $\tau - 1 \geq t+1$. By the first remark after Proposition 9.4, $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p} E_{11} \in S_0$. In fact, this matrix belongs to S_2 since $p^\beta(x \pm \frac{1}{p}) = x' \pm p^{\beta-1}$ is a unit. Hence by Proposition 9.3, $I_{S_2} = 0$ if $p \nmid ac$. If $p|ac$, then $p \nmid b$. In this case, $(p^\alpha x') = (p^t)$, $p^t | p^{\beta-1} y'$, and $\tau - 1 \geq t+1$. By the first remark after Proposition 9.4, $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p} E'_{12} \in S_0$. In fact, the matrix belongs to S_2 since $p^\alpha(y \pm \frac{1}{p}) = y' \pm p^{\alpha-1} \in p\mathbf{Z}_p$. By Proposition 9.3, $I_{S_2} = 0$. The proof that $I_{S_3} = 0$ is similar.

For the integral over S_1, note that the validity of (9.11) and (9.13) depends only on $x', y', z' \pmod{p^{\tau'}}$, which also means that $S_1 = S_1 + S_2(\mathbf{Z}_p)$ (where S_2 here denotes the symmetric matrices). Hence, writing $\mathtt{x}, \mathtt{y}, \mathtt{z}$ for the congruence classes of $x', y', z' \mod p^{\tau'}$,

$$I_{A,p} = I_{S_1} = \int_{S_1} \theta_p(\mathrm{tr}(\sigma_U S)) dS$$

$$= \sum_{\substack{\mathtt{x},\mathtt{y},\mathtt{z} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*, \\ \mathtt{xz} \equiv \mathtt{y}^2 \pmod{p^{\tau'}}}} \int_{\frac{\mathtt{z}}{p^{\tau'}} + \mathbf{Z}_p} \int_{\frac{\mathtt{y}}{p^{\tau'}} + \mathbf{Z}_p} \int_{\frac{\mathtt{x}}{p^{\tau'}} + \mathbf{Z}_p} \theta_p(ax+by+cz)\, dxdydz$$

$$= \sum_{\mathtt{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{b\mathtt{y}}{p^{\tau'}}) \left(\sum_{\substack{\mathtt{x},\mathtt{z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathtt{xz} \equiv \mathtt{y}^2 \pmod{p^{\tau'}}}} e(-\frac{a\mathtt{x} + c\mathtt{z}}{p^{\tau'}}) \right)$$

$$= \sum_{\mathtt{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{b\mathtt{y}}{p^{\tau'}}) \left(\sum_{\substack{\mathtt{x},\mathtt{z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathtt{xz} \equiv 1 \pmod{p^{\tau'}}}} e(-\frac{\mathtt{y}(a\mathtt{x} + c\mathtt{z})}{p^{\tau'}}) \right)$$

$$(9.14) \qquad = \sum_{\substack{\mathtt{x},\mathtt{z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathtt{xz} \equiv 1 \pmod{p^{\tau'}}}} \left(\sum_{\mathtt{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{\mathtt{y}(a\mathtt{x} + c\mathtt{z} + b)}{p^{\tau'}}) \right).$$

We remark that the computation here is valid for $\tau' \geq 1$. The sum over $\mathtt{y}$ can be evaluated using (9.20) below. □

Proposition 9.11. *Suppose α, β, t, τ satisfy Proposition 9.7 (4), i.e. $\tau = 2\tau'$ is even, $\alpha = \beta = \tau'$, and $t = \tau' - 1$. Suppose further that $\tau' \geq 2$. Then the integral $I_{A,p}$ is given by* (9.18) *below.*

Proof. In this case, (9.11) and (9.10) are equivalent to (9.11) and

$$(x'z' - {y'}^2) = (p^{\tau'-1}), \tag{9.15}$$

i.e.,

$$x'z' \equiv {y'}^2 \pmod{p^{\tau'-1}} \tag{9.16}$$

but

$$x'z' \not\equiv {y'}^2 \pmod{p^{\tau'}} \tag{9.17}$$

As in the previous proof, we integrate over S_1, S_2 and S_3. We first show that $I_{S_2} = I_{S_3} = 0$. Suppose $\left(\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right) \in S_2$, so $x' \in \mathbf{Z}_p^*$, $p|y'$, and $p|z'$. Then the conditions of Proposition 9.4(i) are satisfied, so $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E_{11} \in S_0$. Since $p^\beta(x + \frac{1}{p}) = x' + p^{\beta-1}$ is a unit, this matrix in fact belongs to S_2. By Proposition 9.3, $I_{S_2} = 0$, assuming $p \nmid ac$. On the other hand, if $p|ac$, then $p \nmid b$. The hypotheses to Proposition 9.4(ii) are satisfied, so $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \pm \frac{1}{p}E'_{12} \in S_2$, and once again Proposition 9.3 gives $I_{S_2} = 0$. The proof that $I_{S_3} = 0$ is similar.

For the integral over S_1, just as in the proof of the previous proposition, we have

$$I_{A,p} = I_{S_1} = \int_{S_1} \theta_p(\operatorname{tr} \sigma_U S) dS$$

$$= \sum_{\substack{\mathrm{x,y,z} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*, \\ \mathrm{xz} \equiv \mathrm{y}^2 \pmod{p^{\tau'-1}},\ \mathrm{xz} \not\equiv \mathrm{y}^2 \pmod{p^{\tau'}}}} \int_{\frac{\mathrm{z}}{p^{\tau'}} + \mathbf{Z}_p} \int_{\frac{\mathrm{y}}{p^{\tau'}} + \mathbf{Z}_p} \int_{\frac{\mathrm{x}}{p^{\tau'}} + \mathbf{Z}_p} \theta_p(ax + by + cz) dx dy dz$$

$$= \sum_{\mathrm{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{b\mathrm{y}}{p^{\tau'}}) \left(\sum_{\substack{\mathrm{x,z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathrm{xz} \equiv \mathrm{y}^2 \pmod{p^{\tau'-1}}, \mathrm{xz} \not\equiv \mathrm{y}^2 \pmod{p^{\tau'}}}} e(-\frac{a\mathrm{x} + c\mathrm{z}}{p^{\tau'}}) \right)$$

$$= \sum_{\mathrm{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{b\mathrm{y}}{p^{\tau'}}) \left(\sum_{\substack{\mathrm{x,z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathrm{xz} \equiv 1 \pmod{p^{\tau'-1}}, \mathrm{xz} \not\equiv 1 \pmod{p^{\tau'}}}} e(-\frac{\mathrm{y}(a\mathrm{x} + c\mathrm{z})}{p^{\tau'}}) \right)$$

$$= \sum_{\substack{\mathrm{x,z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathrm{xz} \equiv 1 \pmod{p^{\tau'-1}}, \mathrm{xz} \not\equiv 1 \pmod{p^{\tau'}}}} \left(\sum_{\mathrm{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{\mathrm{y}(a\mathrm{x} + c\mathrm{z} + b)}{p^{\tau'}}) \right) \tag{9.18}$$

$$= \sum_{h=1}^{p-1} \sum_{\substack{\mathrm{x,z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathrm{xz} \equiv 1 + hp^{\tau'-1} \pmod{p^{\tau'}}}} \left(\sum_{\mathrm{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{\mathrm{y}(a\mathrm{x} + c\mathrm{z} + b)}{p^{\tau'}}) \right).$$

Once again, this computation is valid for $\tau' \geq 1$. We remark that (9.18) can be rewritten as

$$\sum_{\substack{\mathrm{x,z} \in \mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathrm{xz} \equiv 1 \pmod{p^{\tau'-1}}}} \left(\sum_{\mathrm{y} \in (\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(-\frac{\mathrm{y}(a\mathrm{x} + c\mathrm{z} + b)}{p^{\tau'}}) \right) - (9.14). \tag{9.19}$$

□

Corollary 9.12. *Suppose $\beta - 1 \leq t$ and $\tau \geq 5$. Then $I_{A,p} \ll_p p^{3\tau/4}$.*

Proof. By Propositions 9.7-9.11, we may assume that $\tau = 2\tau'$ is even, and $I_{A,p}$ is given by (9.14) or (9.18). Recall the formula for the Ramanujan sum

$$\sum_{\mathtt{y}\in(\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p)^*} e(\frac{\mathtt{y}\ell}{p^{\tau'}}) = \begin{cases} p^{\tau'} - p^{\tau'-1} & \text{if } p^{\tau'}|\ell, \\ -p^{\tau'-1} & \text{if } p^{\tau'-1}\|\ell, \\ 0 & \text{if } p^{\tau'-1}\nmid \ell \end{cases} \tag{9.20}$$

([Hua, Theorem 4.3]). Thus the summation over $\mathtt{y}$ in (9.14) is nonzero only if

$$a\mathtt{x} + c\mathtt{z} + b \equiv 0 \pmod{p^{\tau'-1}}. \tag{9.21}$$

Since $\mathtt{xz} \equiv 1 \mod p^{\tau'}$, this is equivalent to

$$a\mathtt{x}^2 + b\mathtt{x} + c \equiv 0 \mod p^{\tau'-1}, \tag{9.22}$$

and also to

$$c\mathtt{z}^2 + b\mathtt{z} + a \equiv 0 \mod p^{\tau'-1}. \tag{9.23}$$

Suppose $p|a$ and $p|c$, so that $p \nmid b$. Then (9.21) has no solution, so the summation is zero. If $p \nmid a$ (resp. $p \nmid c$), then the number of $\mathtt{x} \mod p^{\tau'}$ satisfying (9.22) (resp. $\mathtt{z} \mod p^{\tau'}$ satisfying (9.23)) is $\ll pp^{(\tau'-1)/2} \ll_p p^{\tau'/2}$ (cf. [KL3, Lemma 9.6]). Now applying (9.20), we see that (9.14) is $O(p^{3\tau'/2})$.

Similarly, (9.18) is

$$\ll \sum_{h=1}^{p-1} \sum_{\substack{\mathtt{x},\mathtt{z}\in\mathbf{Z}_p/p^{\tau'}\mathbf{Z}_p \\ \mathtt{xz}\equiv 1+hp^{\tau'-1} \pmod{p^{\tau'}}, \\ a\mathtt{x}+c\mathtt{z}+b\equiv 0 \pmod{p^{\tau'-1}}}} p^{\tau'} \ll p^2 p^{\frac{\tau'-1}{2}} p^{\tau'} \ll p^{3\tau'/2}. \qquad \square$$

9.3. **Proof of Proposition 9.1.** For $p \in \mathbf{S}$, we require a bound for the local integral $I_{A,p}$ given in (9.5). We continue to use the notation of Section 9.1. Thus, $\tau = r_p$, $\lambda(p) = \mathrm{diag}(1, p^t, p^\tau, p^{\tau-t})$ for $0 \le t \le \tau/2$, $D = \mathrm{diag}(p^\alpha, p^\beta)$ for $0 \le \alpha \le \beta$ with $\alpha + \beta = \tau$, and $\sigma_U = {}^tU\sigma_1 U = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$.

By (3.9)

$$\rho = \frac{3}{2}e_0 - 2e_1 - e_2,$$

so

$$\langle \lambda, \rho \rangle = \frac{3}{2}\ell_0 - 2\ell_1 - \ell_2 = \frac{3}{2}\tau - t.$$

To prove Proposition 9.1, we must show that for some $\varepsilon > 0$,

$$|I_{A,p}(c_\lambda)| \ll p^{(1-\varepsilon)\langle\lambda,\rho\rangle - \varepsilon\tau} = p^{(1-\varepsilon)(\frac{3}{2}\tau - t) - \varepsilon\tau}. \tag{9.24}$$

By the trivial bound (7.13),

$$|I_{A,p}| \le p^{2\alpha+\beta} = p^{\alpha+\tau}.$$

Therefore (9.24) is certainly satisfied when

$$\alpha + \tau \le (1-\varepsilon)(\frac{3}{2}\tau - t) - \varepsilon\tau, \tag{9.25}$$

or equivalently,

$$\alpha + (1-\varepsilon)t \le \frac{1-5\varepsilon}{2}\tau.$$

Note that $(1-\varepsilon)(\frac{3}{2}\tau - t) - \varepsilon\tau$ is a decreasing function of ε. Therefore if (9.24) holds for some particular $\varepsilon = \varepsilon_0$, then it holds for all smaller positive ε. For concreteness, we will verify it for $\varepsilon_0 = 0.01$, in which case the above inequality takes the form

$$\alpha + 0.99t \leq 0.475\tau. \tag{9.26}$$

For any given value of τ, there are only finitely many permissible values for t, α and β, so the associated integral is bounded by a constant depending only on τ. Therefore we may assume that

$$\tau \geq 7. \tag{9.27}$$

If $p \nmid b$ (resp. $p \nmid ac$) and the conditions of Corollary 9.5 (resp. Corollary 9.6) hold, the integral vanishes and the desired bound is trivially satisfied.

Suppose condition (iii) of either Corollary 9.5 or Corollary 9.6 fails. Then by Corollary 9.12,

$$I_{A,p} \ll p^{3\tau/4} < p^{(1-2\varepsilon)\tau}$$

when $\varepsilon < 1/8$. If $t = \tau/2$ (as is the case in Proposition 9.10), then

$$p^{(1-2\varepsilon)\tau} = p^{(1-\varepsilon)(\frac{3}{2}\tau - t) - \varepsilon\tau}.$$

If $t = \tau/2 - 1$ (as is the case in Proposition 9.11), then

$$p^{(1-2\varepsilon)\tau} \ll p^{(1-2\varepsilon)\tau + (1-\varepsilon)} = p^{(1-\varepsilon)(\frac{3}{2}\tau - t) - \varepsilon\tau}.$$

Either way, we obtain the desired bound for $I_{A,p}$ when $\varepsilon < 1/8$.

Suppose condition (i) of either Corollary 9.5 or Corollary 9.6 fails. Then (using $\alpha \leq \beta$ in the first case) $\alpha \leq 1$. By (9.7), $t \leq 2$. Hence by (9.27),

$$\alpha + 0.99t < 3 < 3.325 = 0.475 \times 7 \leq 0.475\tau,$$

so (9.26) is satisfied in this case, and the desired bound holds.

Suppose condition (ii) of either Corollary 9.5 or Corollary 9.6 fails. Then $\tau - 2 \leq t \leq \frac{\tau}{2}$, which means that $\tau \leq 4$, contradicting (9.27).

This proves (9.24) and hence Proposition 9.1.

Appendix A. Discrete series matrix coefficients for GSp($2n$)

Here we explicitly compute certain discrete series matrix coefficients for $\mathrm{GSp}_{2n}(\mathbf{R})$ using ideas of Harish Chandra. Our main references for the background material are [AS] and [Kn].

A.1. Root System.

The Lie algebra of $\mathrm{Sp}_{2n}(\mathbf{R})$ is

$$\begin{aligned}\mathfrak{g} &= \{X |\, JX + {}^tXJ = 0\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbf{R}) |\, A = -\,{}^tD,\, B = {}^tB,\, C = {}^tC \right\}.\end{aligned}$$

We have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in M_{2n}(\mathbf{R}) |\, A = -\,{}^tA, B = {}^tB \right\}$$

is the Lie algebra of K,[†]

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in M_{2n}(\mathbf{R}) | \, A = {}^tA, B = {}^tB \right\}.$$

Let $\mathfrak{h} \subseteq \mathfrak{k}_{\mathbf{C}}$ denote the real subspace consisting of all matrices of the form

$$\begin{pmatrix} & & & a_1 & & & \\ & & & & a_2 & & \\ & & & & & \ddots & \\ & & & & & & a_n \\ -a_1 & & & & & & \\ & -a_2 & & & & & \\ & & \ddots & & & & \\ & & & -a_n & & & \end{pmatrix}, \tag{A.1}$$

for $a_j \in i\mathbf{R}$ ($i^2 = -1$). Then $\mathfrak{h}$ is a compact Cartan subalgebra of $\mathfrak{g}_{\mathbf{C}}$. For each $j = 1, \ldots, n$, define a linear form e_j on $\mathfrak{h}_{\mathbf{C}}$ by taking the above matrix to ia_j. Let $\Delta = \Delta(\mathfrak{h}_{\mathbf{C}}, \mathfrak{g}_{\mathbf{C}}) \subseteq \mathfrak{h}_{\mathbf{C}}^*$ be the set of roots. One finds that

$$\Delta = \{\pm 2e_j, \pm(e_j + e_k), e_j - e_k | \, j \neq k\}.$$

The compact roots (i.e. those whose root spaces belong to $\mathfrak{k}_{\mathbf{C}}$) are

$$\Delta_K = \{e_j - e_k | \, j \neq k\}.$$

Let $\Delta_{nc} = \Delta - \Delta_K$ denote the set of noncompact roots. We fix the inner product in Δ determined by

$$\langle e_i, e_j \rangle = \delta_{ij}.$$

We set $\varepsilon_j = -e_j$, and fix the following ordered basis for $\mathfrak{h}_{\mathbf{C}}^*$:

$$\{\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n, \ \varepsilon_n - \varepsilon_{n-1}, \ \varepsilon_{n-1} - \varepsilon_{n-2}, \ \ldots, \ \varepsilon_2 - \varepsilon_1\}.$$

This determines a good ordering of Δ (i.e. $\Delta_{nc}^+ > \Delta_K^+$) in which $\Delta^+ = \Delta_{nc}^+ \cup \Delta_K^+$ is given by

$$\left.\begin{array}{rl} 2\varepsilon_j, & 1 \leq j \leq n \\ \varepsilon_j + \varepsilon_k, & 1 \leq j < k \leq n \end{array}\right\} \Delta_{nc}^+$$
$$\left.\begin{array}{rl} \varepsilon_k - \varepsilon_j, & 1 \leq j < k \leq n \end{array}\right\} \Delta_K^+.$$

Then

$$\delta_G = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \varepsilon_1 + 2\varepsilon_2 + \ldots + n\varepsilon_n. \tag{A.2}$$

Remark: The positive system (here denoted ${\Delta'}^+$) considered in [AS] is the one used in Chapter IX of [Kn]. The relationship to the present system is ${\Delta'_K}^+ = \Delta_K^+$ and ${\Delta'_{nc}}^+ = -\Delta_{nc}^+$. In other words, $\Delta^+ = -w_K({\Delta'}^+)$, where w_K is the element of the Weyl group W_K taking Δ_K^+ to $-\Delta_K^+$. (See Remark (1) after Theorem 9.20 in [Kn].)

For each root $\alpha \in \Delta$, let $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{\mathbf{C}}$ be the associated root space. Define the following subalgebras of $\mathfrak{g}_{\mathbf{C}}$:

$$\mathfrak{p}^+ = \bigoplus_{\alpha \in \Delta_{nc}^+} \mathfrak{g}_\alpha = \left\{ \begin{pmatrix} A & -iA \\ -iA & -A \end{pmatrix} | \, A = {}^tA \in M_n(\mathbf{C}) \right\},$$

and

$$\mathfrak{p}^- = \bigoplus_{\alpha \in \Delta_{nc}^+} \mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} A & iA \\ iA & -A \end{pmatrix} | \, A = {}^tA \in M_n(\mathbf{C}) \right\}.$$

[†]In this appendix, K denotes the compact subgroup (3.3) which was denoted K_∞ elsewhere in the paper.

The corresponding analytic subgroups of $\mathrm{Sp}_{2n}(\mathbf{C})$ are

$$P^+ = \exp(\mathfrak{p}^+) = \left\{ \begin{pmatrix} I_n + A & -iA \\ -iA & I_n - A \end{pmatrix} \mid A = {}^tA \in M_n(\mathbf{C}) \right\}$$

and

$$P^- = \exp(\mathfrak{p}^-) = \left\{ \begin{pmatrix} I_n + A & iA \\ iA & I_n - A \end{pmatrix} \mid A = {}^tA \in M_n(\mathbf{C}) \right\}.$$

We also have

$$K_{\mathbf{C}} = \exp(\mathfrak{k}_{\mathbf{C}}) = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{GL}_{2n}(\mathbf{C}) \mid (A + iB)\ {}^t(A - iB) = I_n \right\}.$$

Since A and B are complex, the condition on $(A + iB)$ is equivalent to $A + iB \in \mathrm{GL}_n(\mathbf{C})$, reflecting the fact that $\mathrm{U}(n, \mathbf{C}) \cong \mathrm{GL}_n(\mathbf{C})$.

A.2. **Realization in** $\mathrm{SU}(n, n)$**.** Recall that

$$\mathrm{SU}(n, n) = \left\{ g \in \mathrm{SL}_{2n}(\mathbf{C}) \mid {}^t\overline{g} \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} g = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} \right\}.$$

Define

$$G' = \{ g \in \mathrm{SU}(n, n) \mid {}^tgJg = J \}.$$

One can show that

$$G' = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathrm{SL}_{2n}(\mathbf{C}) \;\middle|\; \begin{array}{c} {}^t\overline{\alpha}\alpha - {}^t\beta\overline{\beta} = I_n \\ {}^t\overline{\beta}\alpha = {}^t\alpha\overline{\beta} \end{array} \right\}.$$

Let $\tau = \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}$. Then the map

$$g \mapsto g' = \tau^{-1} g \tau$$

is an isomorphism from $\mathrm{Sp}_{2n}(\mathbf{R})$ into G'. For any object O associated to $\mathrm{Sp}_{2n}(\mathbf{R})$, we let O' denote the corresponding object for G'. Writing $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have

$$g' = \frac{1}{2} \begin{pmatrix} (A + D) + i(B - C) & (B + C) + i(A - D) \\ (B + C) - i(A - D) & (A + D) - i(B - C) \end{pmatrix}. \tag{A.3}$$

Taking $g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K$, we see that $g' = \begin{pmatrix} A + Bi & \\ & A - Bi \end{pmatrix}$, where $A + Bi$ is unitary. Thus

$$K' = \tau^{-1} K \tau = \left\{ \begin{pmatrix} \alpha & \\ & {}^t\alpha^{-1} \end{pmatrix} \mid \alpha \in \mathrm{U}(n) \right\}.$$

Likewise,

$$K'_{\mathbf{C}} = \tau^{-1} K_{\mathbf{C}} \tau = \left\{ \begin{pmatrix} \alpha & \\ & {}^t\alpha^{-1} \end{pmatrix} \mid \alpha \in \mathrm{GL}_n(\mathbf{C}) \right\}$$

$$P'^+ = \tau^{-1} P^+ \tau = \left\{ \begin{pmatrix} I_n & O \\ -2iA & I_n \end{pmatrix} \mid A = {}^tA \in M_n(\mathbf{C}) \right\}$$

and

$$P'^- = \tau^{-1} P^- \tau = \left\{ \begin{pmatrix} I_n & 2iA \\ O & I_n \end{pmatrix} \mid A = {}^tA \in M_n(\mathbf{C}) \right\}.$$

A.3. **Holomorphic discrete series.** We recall without proof some properties of the holomorphic discrete series for the group $G = \mathrm{Sp}_{2n}(\mathbf{R})$. This material is due to Harish-Chandra [HC]. We follow the exposition in Chapter VI of [Kn]. Suppose $\lambda \in \mathfrak{h}_{\mathbf{C}}^*$ is analytically integral (i.e. $\lambda(H) \in 2\pi i\mathbf{Z}$ whenever $\exp(H) = 1$) and dominant with respect to K (i.e. $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in \Delta_K^+$). Let (Φ_λ, V) denote the irreducible unitary representation of K with highest weight λ. Let $v_\lambda \in V$ be a highest weight unit vector (unique up to unitary scaling).

Extend Φ_λ to a holomorphic representation of $K_{\mathbf{C}}$. For $g \in G$, let $\mu(g) \in K_{\mathbf{C}}$ denote the middle component in the Harish-Chandra decomposition

$$G \subseteq P^+ K_{\mathbf{C}} P^-$$

([Kn, Theorem 6.3]). Define

$$\psi_\lambda(g) = \left\langle \Phi_\lambda(\mu(g))^{-1} v_\lambda, v_\lambda \right\rangle_V .$$

Then under the condition

$$\langle \lambda + \delta_G, \alpha \rangle < 0 \quad \text{for all} \quad \alpha \in \Delta_{nc}^+, \tag{A.4}$$

ψ_λ is a nontrivial square-integrable function on G ([Kn, Lemma 6.9]), and its translates under the left regular representation generate an irreducible square-integrable representation (π_λ, V_λ) of G ([Kn, Theorem 6.6]). Furthermore, by [Kn, p. 160 (6)],

$$\langle \pi_\lambda(g)\psi_\lambda, \psi_\lambda \rangle_{L^2} = \psi_\lambda(g^{-1}) \|\psi_\lambda\|^2.$$

To each $v \in V$ we associate the function $\left\langle \Phi_\lambda(\mu(g))^{-1} v, v_\lambda \right\rangle \in V_\lambda$. This defines a K-equivariant embedding $V \to V_\lambda$, and hence Φ_λ occurs as a K-type in π_λ, with highest weight vector ψ_λ. In fact, this K-type occurs with multiplicity one and $w_\lambda = \frac{\psi_\lambda}{\|\psi_\lambda\|} \in V_\lambda$ is a highest weight unit vector ([Kn, p. 160 (5)]). Our aim is to compute the matrix coefficient

$$\langle \pi_\lambda(g) w_\lambda, w_\lambda \rangle = \psi_\lambda(g^{-1}) = \left\langle \Phi_\lambda(\mu(g^{-1})^{-1}) v_\lambda, v_\lambda \right\rangle_V . \tag{A.5}$$

We will use the realization of G in $\mathrm{SU}(n,n)$ since it facilitates the computation of $\mu(g)$. Let $g' = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in G'$. Then the Harish-Chandra decomposition is given explicitly by

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} I_n & O \\ \overline{\beta}\alpha^{-1} & I_n \end{pmatrix} \begin{pmatrix} \alpha & O \\ O & {}^t\alpha^{-1} \end{pmatrix} \begin{pmatrix} I_n & \alpha^{-1}\beta \\ O & I_n \end{pmatrix}. \tag{A.6}$$

To verify this decomposition, note that the lower right corner on the right-hand side is ${}^t\alpha^{-1} + \overline{\beta}\alpha^{-1}\beta$. By the fact that ${}^t\alpha\overline{\alpha} = I_n + \overline{{}^t\beta}\beta$,

$$\overline{\alpha} = {}^t\alpha^{-1} + {}^t\alpha^{-1}\overline{{}^t\beta}\beta.$$

Also $\overline{{}^t\beta}\alpha = {}^t\alpha\overline{\beta} \implies {}^t\alpha^{-1} = \overline{\beta}\alpha^{-1}\overline{{}^t\beta}^{-1}$. Substituting this into the second term above, we see that the lower right-hand corner is equal to $\overline{\alpha}$ as needed.

Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbf{R})$, and let $g' = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in G'$ as in (A.3). By (A.6), we see that $\mu(g') = \begin{pmatrix} \alpha & O \\ O & {}^t\alpha^{-1} \end{pmatrix}$. Now $g'^{-1} = \begin{pmatrix} {}^t\overline{\alpha} & -{}^t\beta \\ -\overline{{}^t\beta} & {}^t\alpha \end{pmatrix}$, so

$$\mu(g'^{-1}) = \begin{pmatrix} {}^t\overline{\alpha} & O \\ O & \overline{\alpha}^{-1} \end{pmatrix}. \tag{A.7}$$

A.4. **Explicit formula for the matrix coefficient.** Although it is possible to compute the matrix coefficient of π_λ explicitly for any λ (at least when $n = 2$), for simplicity, we have chosen here to treat just the case where $\dim \Phi_\lambda = 1$. This discussion is valid for any $n \geq 2$.

For $\mathtt{k} \in \mathbf{Z}$, let $\Phi_\mathtt{k}$ be the character of $K = \mathrm{U}(n)$ defined by

$$\Phi_\mathtt{k}(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}) = \det(A + Bi)^\mathtt{k}. \tag{A.8}$$

Its holomorphic extension to $\mathrm{U}_n(\mathbf{C}) = \mathrm{GL}_n(\mathbf{C})$ is given by the same formula. The (unique) weight of this character is

$$\lambda_\mathtt{k} = -\mathtt{k}(\varepsilon_1 + \cdots + \varepsilon_n). \tag{A.9}$$

This weight satisfies condition (A.4) exactly when $\mathtt{k} > n$.

Proposition A.1. *The character $\Phi_\mathtt{k} = \det^\mathtt{k}$ arises as the minimal K-type of a holomorphic discrete series representation $\pi_\mathtt{k}^+$ of $\mathrm{Sp}_{2n}(\mathbf{R})$ if and only if $\mathtt{k} > n$. If this holds and $w_\mathtt{k}$ is a unit vector of weight $\lambda_\mathtt{k}$, then for any $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbf{R})$ we have*

$$\langle \pi_\mathtt{k}^+(g) w_\mathtt{k}, w_\mathtt{k} \rangle = \frac{2^{n\mathtt{k}}}{\det(A + D + i(-B + C))^\mathtt{k}}.$$

Proof. By (A.5) and (A.7),

$$\langle \pi_\mathtt{k}^+(g) w_\mathtt{k}, w_\mathtt{k} \rangle = \Phi_\mathtt{k}(\mu(g^{-1}))^{-1} = \det({}^t\overline{\alpha})^{-\mathtt{k}}.$$

By (A.3), if $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$,

$$\alpha = \frac{1}{2}((A + D) + i(B - C)). \tag{A.10}$$

The given formula now follows. $\square$

Corollary A.2. *For $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_{2n}(\mathbf{R})$, define $f_\mathtt{k}^+(g) = \overline{\langle \pi_\mathtt{k}(g) w_\mathtt{k}, w_\mathtt{k} \rangle}$. Then*

$$f_\mathtt{k}^+(g) = \frac{2^{n\mathtt{k}}}{\det(A + D + i(B - C))^\mathtt{k}}. \tag{A.11}$$

Corollary A.3. *We have*

$$|f_\mathtt{k}^+(g)| = \frac{2^{n\mathtt{k}}}{\det(2I_n + A\,{}^tA + B\,{}^tB + C\,{}^tC + D\,{}^tD + i(A\,{}^tC - C\,{}^tA + B\,{}^tD - D\,{}^tB))^{\mathtt{k}/2}}.$$

Proof. For $\alpha = A + D + i(B - C)$, $|\det \alpha|^2 = \det \alpha \det {}^t\overline{\alpha} = \det \alpha\, {}^t\overline{\alpha}$. Expand this using the relations given in (3.1) and (3.2). The corollary then follows immediately. $\square$

With the formula for the matrix coefficient in hand, we can compute its L^p-norms. The formulas above and the calculations below closely parallel those for $\mathrm{GL}_2(\mathbf{R})$ given in [KL1, §14].

Proposition A.4. *For any real number $\ell > 0$, the function $|f_\mathtt{k}^+|^\ell$ is integrable over $G = \mathrm{Sp}_{2n}(\mathbf{R})$ if and only if $\ell\mathtt{k} > 2n$. If this condition holds, then with Haar measure normalized as in the proof below,*

$$\int_G |f_\mathtt{k}^+(g)|^\ell dg = \frac{2^{n(n+1)} \prod_{j=1}^{n-1} j!}{\prod_{1 \leq i \leq j \leq n} (\ell\mathtt{k} - (i + j))}.$$

Proof. The matrix coefficient $f_{\mathbf{k}}^+$ is bi-K-invariant, so it is convenient to use the Cartan decomposition $G = KA^+K$, where

$$A^+ = \{a = \mathrm{diag}(a_1, \dots, a_n, a_1^{-1}, \dots, a_n^{-1}) \,|\, 1 < a_1 < a_2 < \cdots < a_n\}.$$

We may view Δ^+ as the set of positive roots relative to the action of the diagonal subgroup on $\mathfrak{g}$, and $A^+ = \exp(\mathfrak{a}^+)$, where $\mathfrak{a}^+$ is the positive Weyl chamber. By a standard integration formula ([vdB, Lemma 4.2]), when dg is suitably normalized we have

$$\int_G |f_{\mathbf{k}}^+(g)|^\ell dg = \int_{K\times A^+\times K} |f_{\mathbf{k}}^+(k_1 a k_2)|^\ell \prod_{\alpha\in\Delta^+} (a^\alpha - a^{-\alpha})\, dk_1 da\, dk_2,$$

where the Haar measure of the compact group K is taken to be 1. Note that $(a^\alpha - a^{-\alpha}) > 0$ by the definition of A^+. We now change notation and write $a = \mathrm{diag}(a_1, \dots, a_n)$. Using Corollary A.3, the above is

$$= 2^{n\ell\mathbf{k}} \int_{\substack{a\in \mathrm{GL}_n(\mathbf{R}) \text{ diagonal},\\ 1<a_1<\cdots<a_n}} \det(2I_n + a^2 + a^{-2})^{-\ell\mathbf{k}/2} \prod_{\alpha\in\Delta^+} (a^\alpha - a^{-\alpha}) da$$

$$= 2^{n\ell\mathbf{k}} \int_1^\infty \int_{a_1}^\infty \cdots \int_{a_{n-1}}^\infty \prod_{d=1}^n (2 + a_d^2 + a_d^{-2})^{-\ell\mathbf{k}/2} (a_d^2 - a_d^{-2})$$

$$\times \prod_{1\le i<j\le n} (a_i a_j - a_i^{-1} a_j^{-1})(a_j a_i^{-1} - a_j^{-1} a_i) \frac{da_n}{a_n} \cdots \frac{da_1}{a_1}.$$

To ease notation below, set $\kappa = \ell\mathbf{k}$. Then letting $u_j = 2 + a_j^2 + a_j^{-2}$, the above is

$$= 2^{n\kappa - n} \int_4^\infty \int_{u_1}^\infty \cdots \int_{u_{n-1}}^\infty \prod_{d=1}^n u_d^{-\kappa/2} \prod_{1\le i<j\le n} (u_j - u_i)\, du_n \cdots du_1. \tag{A.12}$$

This can be evaluated using the Selberg integral, but we have chosen to give a self-contained treatment since it does not take much more space to do so. Observe that

$$\prod_{i<j} (u_j - u_i) = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) u_1^{\sigma(1)-1} \cdots u_n^{\sigma(n)-1}.$$

Hence (A.12) becomes

$$2^{n\kappa-n} \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) \int_4^\infty \int_{u_1}^\infty \int_{u_2}^\infty \cdots \int_{u_{n-1}}^\infty u_1^{\sigma(1)-1-\kappa/2} \cdots u_n^{\sigma(n)-1-\kappa/2} du_n \cdots du_1$$

$$= 2^{n\kappa-n} \sum_{\sigma\in S_n} \frac{\mathrm{sgn}(\sigma)(-1)^n 4^{\sigma(1)+\sigma(2)+\cdots+\sigma(n)-n\kappa/2}}{\prod_{j=1}^n \big(\sigma(n) + \sigma(n-1) + \cdots + \sigma(n-j+1) - j\kappa/2\big)}$$

$$= 2^{n^2} \sum_{\sigma\in S_n} \frac{\mathrm{sgn}(\sigma)}{\big(\kappa/2 - \sigma(n)\big)\big(\kappa - \sigma(n) - \sigma(n-1)\big) \cdots \big(n\kappa/2 - \sigma(n) - \cdots - \sigma(1)\big)},$$

provided $\kappa > 2n$ (otherwise the integral diverges). Replace σ by $\sigma\tau$, where $\tau(i) = n - i + 1$ for all $1 \le i \le n$. The above is then

$$= 2^{n^2} \sum_{\sigma\in S_n} \frac{\mathrm{sgn}(\sigma\tau)}{\big(\kappa/2 - \sigma(1)\big)\big(\kappa - \sigma(1) - \sigma(2)\big) \cdots \big(n\kappa/2 - \sigma(1) - \cdots - \sigma(n)\big)}.$$

Applying Lemma A.5 below with $b_i = \kappa/2 - i$, the above is

$$= 2^{n^2+n} \frac{\mathrm{sgn}(\tau) \prod_{i<j} (i-j)}{\prod_{i\le j} (\kappa - (i+j))} = 2^{n(n+1)} \frac{\prod_{i<j} (j-i)}{\prod_{i\le j} (\kappa - (i+j))},$$

by the fact that $\operatorname{sgn}(\tau) = (-1)^{\binom{n}{2}}$ (both are 1 iff $n \equiv 0, 1 \mod 4$). The proposition now follows. $\square$

Lemma A.5. *Let F be a field of characteristic 0, and let $b_1, \ldots, b_n$ be indeterminate variables. Then in the field $F(b_1, \ldots, b_n)$ of rational functions,*

$$\sum_{\sigma \in S_n} \frac{\operatorname{sgn}(\sigma)}{b_{\sigma(1)}(b_{\sigma(1)} + b_{\sigma(2)}) \cdots (b_{\sigma(1)} + \cdots + b_{\sigma(n)})} = 2^n \frac{\prod_{i<j}(b_j - b_i)}{\prod_{i \le j}(b_i + b_j)}. \tag{A.13}$$

Proof. Let $A(b_1, \ldots, b_n)$ denote the left-hand side of (A.13). Then

$$A(b_1, \ldots, b_n) = \sum_{\ell=1}^{n} (-1)^{n-\ell} \frac{A(b_1, \ldots, b_{\ell-1}, b_{\ell+1}, \ldots, b_n)}{b_1 + \cdots + b_n}. \tag{A.14}$$

To see this, write

$$A(b_1, \ldots, b_n) = \frac{1}{b_1 + \cdots + b_n} \sum_{\ell=1}^{n} \sum_{\substack{\sigma \in S_n, \\ \sigma(n) = \ell}} \frac{\operatorname{sgn}(\sigma)}{b_{\sigma(1)}(b_{\sigma(1)} + b_{\sigma(2)}) \cdots (b_{\sigma(1)} + \cdots + b_{\sigma(n-1)})}.$$

Given $\sigma \in S_n$ with $\sigma(n) = \ell$, define $\sigma' \in S_{n-1} \subseteq S_n$ by

$$\sigma'(i) = \begin{cases} \sigma(i) & \text{if } 1 \le \sigma(i) < \ell \\ \sigma(i) - 1 & \text{if } \ell + 1 \le \sigma(i) \le n. \end{cases}$$

Then σ is the composition of σ' with $n - \ell$ transpositions, so $\operatorname{sgn}(\sigma) = (-1)^{n-\ell} \operatorname{sgn}(\sigma')$. Since $\sigma \mapsto \sigma'$ defines a bijection between the set of such σ and S_{n-1}, (A.14) follows.

We may now prove (A.13) by induction on n. The base case $n = 2$ is easy to check by hand. Applying the inductive hypothesis to (A.14),

$$A(b_1, \ldots, b_n) = \frac{2^{n-1}}{b_1 + \cdots + b_n} \sum_{\ell=1}^{n} (-1)^{n-\ell} \frac{\prod_{i<j; i,j \ne \ell}(b_j - b_i)}{\prod_{i \le j; i,j \ne \ell}(b_i + b_j)}.$$

Let

$$B(b_1, \ldots, b_n) = \frac{b_1 + \cdots + b_n}{2^{n-1}} \left(\prod_{1 \le i \le j \le n} (b_i + b_j) \right) A(b_1, \ldots, b_n) \tag{A.15}$$

$$= \sum_{\ell=1}^{n} (-1)^{n-\ell} \left(\prod_{i<j; i,j \ne \ell} (b_j - b_i) \right) \left(\prod_{i=1}^{n} (b_i + b_\ell) \right). \tag{A.16}$$

This is a homogeneous polynomial of degree $\frac{(n-1)(n-2)}{2} + n = \frac{n(n-1)}{2} + 1$. Because

$$A(b_1, \ldots, b_n) = \operatorname{sgn}(\sigma) A(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$$

for all permutations $\sigma \in S_n$, B inherits this property from (A.15). In particular,

$$B(b_1, \ldots, b_n) = -B(b_{\sigma(1)}, \ldots, b_{\sigma(n)}) \tag{A.17}$$

if $\sigma = (i\ j)$ is any 2-cycle. It follows that $B(b_1, \ldots, b_n) = 0$ if $b_i = b_j$ for any $i \ne j$. Hence $\prod_{i<j}(b_i - b_j)$ divides $B(b_1, \ldots, b_n)$, and

$$\frac{B(b_1, \ldots, b_n)}{\prod_{i<j}(b_j - b_i)}$$

is a homogeneous symmetric polynomial of degree $\frac{n(n-1)}{2} + 1 - \frac{n(n-1)}{2} = 1$. Hence it has the form $c(b_1 + \cdots + b_n)$ for some constant c. The monomial $b_n^n \prod_{i=2}^{n-1} b_i^{i-1}$ appears in (A.16) with coefficient 2 (take $\ell = n$), and in $(b_1 + \cdots + b_n)\prod_{i<j}(b_j - b_i)$ with coefficient 1. Therefore $c = 2$, and (A.13) follows. □

A.5. **Integrability.** An irreducible unitary representation π of $G = \mathrm{Sp}_{2n}(\mathbf{R})$ is said to be *integrable* if it has a nonzero matrix coefficient belonging to $L^1(G)$, or equivalently, if *all* of its K-finite matrix coefficients belong to $L^1(G)$. Applying Proposition A.4 with $\ell = 1$, we immediately obtain the following.

Proposition A.6. *The representation $\pi_{\mathtt{k}}^+$ is integrable if and only if $\mathtt{k} > 2n$.*

More generally, let π_λ be the discrete series representation of G (holomorphic or not) with Harish-Chandra parameter λ. Then by a theorem due to Trombi, Varadarajan, Hecht and Schmid, π_λ is integrable if and only if

$$\text{(A.18)} \qquad |\langle \lambda, \beta \rangle| > \tfrac{1}{2} \sum_{\alpha \in \Delta^+} |\langle \alpha, \beta \rangle| \quad \text{for all} \quad \beta \in \Delta_{nc}$$

([TV], [HS]; see also Miličić [Mi]). With notation as in (A.2) and (A.9), the Harish-Chandra parameter of $\pi_{\mathtt{k}}^+$ is

$$\text{(A.19)} \qquad \lambda = \lambda_{\mathtt{k}} + \delta_G = (1-\mathtt{k})\varepsilon_1 + (2-\mathtt{k})\varepsilon_2 + \cdots + (n-\mathtt{k})\varepsilon_n$$

(see Remark (1) after Theorem 9.20 of [Kn]). Note that for any j, ℓ,

$$\text{(A.20)} \qquad |\langle \lambda_{\mathtt{k}} + \delta_G, \varepsilon_j + \varepsilon_\ell \rangle| = |(j+\ell) - 2\mathtt{k}| = 2\mathtt{k} - (j+\ell).$$

Using this, one may easily verify that (A.18) holds for λ exactly when $\mathtt{k} > 2n$, for an alternative proof of Proposition A.6.

A.6. **Formal Degree.** Recall that the formal degree of $\pi = \pi_\lambda$ is the constant $d_\pi > 0$ (depending only on the choice of Haar measure on $G = \mathrm{Sp}_{2n}(\mathbf{R})$) satisfying

$$\int_G |\langle \pi_\lambda(g)v, w\rangle|^2 dg = \frac{\|v\|^2\|w\|^2}{d_\lambda}$$

for all $v, w \in V_\pi$. Applying Proposition A.4 with $\ell = 2$, we immediately find the following.

Proposition A.7. *The formal degree of $\pi_{\mathtt{k}}^+$ is the following polynomial in $\mathtt{k}$ of degree $n + \binom{n}{2} = \frac{n(n+1)}{2}$:*

$$\text{(A.21)} \qquad d_{\mathtt{k}} = a \prod_{1 \le i \le j \le n} (2\mathtt{k} - (i+j)),$$

where a is a nonzero constant depending on dg.

Remarks: (1) Harish-Chandra proved that there exists a choice of Haar measure for which

$$d_\lambda = \prod_{\beta \in \Delta^+} \left| \frac{\langle \lambda + \delta_G, \beta \rangle}{\langle \delta_G, \beta \rangle} \right|$$

for all λ ([HC] §10). If λ is given by (A.19), then by evaluating the above expression explicitly as in (A.20), we obtain an alternative proof of (A.21).
(2) With measure normalized as in the proof of Proposition A.4, $a = (2^{n(n+1)} \prod_{j=1}^{n-1} j!)^{-1}$.
(3) If we adopt the classical normalization of measure, so that

$$d_{\mathtt{k}}^{-1} = \int_G |f_{\mathtt{k}}^+(g)|^2 dg = \int_{\mathcal{H}_n} \left| f_{\mathtt{k}}^+\left(\left(\begin{smallmatrix} I_n & X \\ & I_n \end{smallmatrix}\right) \left(\begin{smallmatrix} Y^{1/2} & \\ & Y^{-1/2} \end{smallmatrix}\right) \right) \right|^2 \frac{dX\,dY}{(\det Y)^{n+1}},$$

then $a = 2^{-n(n+2)}\pi^{-n(n+1)/2}$ ([PSS, §A.1]).

A.7. **Extension of $\pi_{\mathtt{k}}^+$ to GSp_{2n}.** We can extend $\pi_{\mathtt{k}}^+$ to a representation of $\mathrm{GSp}_{2n}(\mathbf{R})$ in the following way. First induce $\pi_{\mathtt{k}}^+$ to the group of symplectic similitudes with multiplier ± 1, namely

$$\mathrm{Sp}_{2n}^{\pm} = \left\langle \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} \right\rangle \mathrm{Sp}_{2n}.$$

Let V^+ be the space of $\pi_{\mathtt{k}}^+$. Then the new space is

$$V = \{f : \mathrm{Sp}_{2n}^{\pm}(\mathbf{R}) \to V^+ |\, f(gx) = \pi_{\mathtt{k}}^+(g)f(x) \text{ for all } g \in \mathrm{Sp}_{2n}(\mathbf{R})\},$$

and $\mathrm{Sp}_{2n}^{\pm}(\mathbf{R})$ acts by right translation. Note that any $f \in V$ is determined by $f(\left(\begin{smallmatrix} I_n & \\ & I_n \end{smallmatrix}\right))$ and $f(\left(\begin{smallmatrix} I_n & \\ & -I_n \end{smallmatrix}\right))$. We identify V^+ with the subspace of $V^{\pm}$ consisting of functions which vanish on $\left(\begin{smallmatrix} I_n & \\ & -I_n \end{smallmatrix}\right)$. Letting V^- denote the space of functions vanishing on the identity element, we have

$$V = V^+ \oplus V^-. \tag{A.22}$$

We make V into a Hilbert space by defining

$$\langle f, h \rangle = \langle f(1), h(1) \rangle_{V^+} + \langle f(\sigma), h(\sigma) \rangle_{V^+}, \tag{A.23}$$

where $\sigma = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}$. Then (A.22) is an orthogonal direct sum.

Denote this representation on V by $\pi_{\mathtt{k}}$. One easily sees that each subspace in (A.22) is stable under $\mathrm{Sp}_{2n}(\mathbf{R})$, and $\pi_{\mathtt{k}}|_{\mathrm{Sp}_{2n}(\mathbf{R})} = \pi_{\mathtt{k}}^+ \oplus \pi_{\mathtt{k}}^-$, where $\pi_{\mathtt{k}}^-$ is also irreducible and square integrable. Let

$$Z^+ = \{ \begin{pmatrix} zI_n & \\ & zI_n \end{pmatrix} | \, z > 0 \}.$$

Then

$$\mathrm{GSp}_{2n} = Z^+ \times \mathrm{Sp}_{2n}^{\pm}.$$

Extend $\pi_{\mathtt{k}}$ to a representation of $\mathrm{GSp}_{2n}(\mathbf{R})$ by requiring Z^+ to act trivially. This is an irreducible square integrable representation, also denoted $\pi_{\mathtt{k}}$. For any $z \in Z(\mathbf{R})$ and $f \in V$, we have

$$\pi_{\mathtt{k}}(z)f(g) = f(\mathrm{sgn}(z)g) = \pi_{\mathtt{k}}^+(\mathrm{sgn}(z))f(g) = \mathrm{sgn}(z)^{n\mathtt{k}} f(g)$$

by (A.8). This shows that the central character of $\pi_{\mathtt{k}}$ is

$$\chi_{\pi_{\mathtt{k}}}(z) = \mathrm{sgn}(z)^{n\mathtt{k}}. \tag{A.24}$$

As before, let $w_{\mathtt{k}} \in V^+$ denote a unit vector of weight $\lambda_{\mathtt{k}}$. Define $\phi_0 \in V$ by

$$\phi_0(\begin{pmatrix} I_n & \\ & I_n \end{pmatrix}) = w_{\mathtt{k}} \quad \text{and} \quad \phi_0(\begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}) = 0.$$

This is a lowest weight vector, spanning the minimal K-type of $\pi_{\mathtt{k}}$, which is the two-dimensional representation $\mathrm{Ind}_K^{K^{\pm}}(\Phi_{\mathtt{k}})$, where $K^{\pm} = K \cup K\left(\begin{smallmatrix} I_n & \\ & -I_n \end{smallmatrix}\right)$.

Proposition A.8. *The representation $\pi_{\mathtt{k}}$ is irreducible, unitary, and square-integrable when $\mathtt{k} > n$. It is integrable exactly when $\mathtt{k} > 2n$, and in this case, the formal degree of $\pi_{\mathtt{k}}$ coincides with that of $\pi_{\mathtt{k}}^+$ given in* (A.21).

Proof. Everything follows more or less immediately from the corresponding properties of $\pi_{\mathtt{k}}^+$. Indeed, define the matrix coefficient

$$f_{\mathtt{k}}(g) = \overline{\langle \pi_{\mathtt{k}}(g)\phi_0, \phi_0 \rangle}. \tag{A.25}$$

Unraveling the definitions, we see that $f_{\mathrm{k}}(g) = f_{\mathrm{k}}^+(g)$ for $g \in \mathrm{Sp}_{2n}(\mathbf{R})$, and $f_{\mathrm{k}}(g) = 0$ if $r(g) < 0$. Using the fact that π_{k} is Z^+-invariant, we have

$$\int_{\mathrm{GSp}_{2n}(\mathbf{R})/Z} |f_{\mathrm{k}}(g)|^\ell dg = \int_{\mathrm{Sp}_{2n}^{\pm}} |f_{\mathrm{k}}(g)|^\ell dg = \int_{\mathrm{Sp}_{2n}} |f_{\mathrm{k}}(g)|^\ell dg = \int_{\mathrm{Sp}_{2n}} |f_{\mathrm{k}}^+(g)|^\ell dg.$$

The assertions now follow from Proposition A.4. □

Theorem A.9. *For* $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_{2n}(\mathbf{R})$,

$$\langle \pi_{\mathrm{k}}(g)\phi_0, \phi_0 \rangle = \begin{cases} \dfrac{r(g)^{\frac{n\mathrm{k}}{2}} 2^{n\mathrm{k}}}{\det(A + D + i(-B + C))^{\mathrm{k}}} & \text{if } r(g) > 0 \\ \\ 0 & \text{if } r(g) < 0. \end{cases}$$

Proof. Let $r = r(g)$. Suppose $r < 0$. Then $\pi_{\mathrm{k}}(g)\phi_0(x) \neq 0 \iff r(x) < 0$, i.e. $\pi_{\mathrm{k}}(g)\phi_0 \in V^-$, so it is orthogonal to $\phi_0 \in V^+$. Thus $\langle \pi_{\mathrm{k}}(g)\phi_0, \phi_0 \rangle = 0$ in this case.

Now suppose $r > 0$. Let

$$h = r^{-1/2} g = \begin{pmatrix} r^{-1/2}A & r^{-1/2}B \\ r^{-1/2}C & r^{-1/2}D \end{pmatrix}.$$

It is easy to see that $r(h) = 1$, i.e. that $h \in \mathrm{Sp}_{2n}(\mathbf{R})$. By definition of π_{k}, $\pi_{\mathrm{k}}(g)\phi_0 = \pi_{\mathrm{k}}(h)\phi_0$. Therefore by (A.23),

$$\begin{aligned} \langle \pi_{\mathrm{k}}(g)\phi_0, \phi_0 \rangle &= \langle \pi_{\mathrm{k}}(h)\phi_0, \phi_0 \rangle = \left\langle \pi_{\mathrm{k}}^+(h) w_{\mathrm{k}}, w_{\mathrm{k}} \right\rangle_{V^+} \\ &= \frac{2^{n\mathrm{k}}}{r^{-n\mathrm{k}/2} \det(A + D + i(-B + C))^{\mathrm{k}}} \end{aligned}$$

by Proposition A.1. □

The following corollaries are easily proven as in §A.4.

Corollary A.10. *For* $f_{\mathrm{k}}(g) = \overline{\langle \pi_{\mathrm{k}}(g)\phi_0, \phi_0 \rangle}$, *if* $r(g) > 0$, *then*

$$f_{\mathrm{k}}(g) = \frac{r(g)^{\frac{n\mathrm{k}}{2}} 2^{n\mathrm{k}}}{\det(A + D + i(B - C))^{\mathrm{k}}}.$$

If $r(g) < 0$, *then* $f_{\mathrm{k}}(g) = 0$.

Corollary A.11. *If* $g \in \mathrm{GSp}_{2n}(\mathbf{R})$ *with* $r(g) > 0$, *then*

$$|f_{\mathrm{k}}(g)| = \frac{r(g)^{\frac{n\mathrm{k}}{2}} 2^{n\mathrm{k}}}{\det(2r(g)I_n + A\,{}^tA + B\,{}^tB + C\,{}^tC + D\,{}^tD + i(A\,{}^tC - C\,{}^tA + B\,{}^tD - D\,{}^tB))^{\mathrm{k}/2}}.$$

In the special case $n = 2$, we can make the above more explicit and provide a convenient upper bound for the matrix coefficient.

Proposition A.12. *Suppose* $n = 2$, *and* $r(g) > 0$. *Then*

$$|f_{\mathrm{k}}(g)| = \frac{r(g)^{\mathrm{k}} 4^{\mathrm{k}}}{(4r(g)^2 + 2r(g)\sum_{i,j} g_{ij}^2 + \sum_{i=3}^8 X_i^2)^{\mathrm{k}/2}},$$

where $g_{i,j}$ *are the entries of* g, *and the* X_i *are the bilinear forms in these entries defined in the proof below. Consequently,*

$$|f_{\mathrm{k}}(g)| \leq \frac{(8r(g))^{\mathrm{k}/2}}{(2r(g) + \sum_{i,j} g_{ij}^2)^{\mathrm{k}/2}}. \tag{A.26}$$

Proof. Write $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Let A_{ij} denote the (i,j)-th entry of A, and likewise for the entries of B, C, D. Then

$$\begin{aligned}
&\det\big(2r(g)I_2 + A\,{}^tA + B\,{}^tB + C\,{}^tC + D\,{}^tD + i(A\,{}^tC - C\,{}^tA + B\,{}^tD - D\,{}^tB)\big)\\
&=(2r(g) + A_{11}^2 + A_{12}^2 + B_{11}^2 + B_{12}^2 + B_{11}^2 + C_{12}^2 + D_{11}^2 + D_{12}^2)\\
&\quad\times(2r(g) + A_{21}^2 + A_{22}^2 + B_{21}^2 + B_{22}^2 + C_{21}^2 + C_{22}^2 + D_{21}^2 + D_{22}^2) - X_1^2 - X_2^2,
\end{aligned} \tag{A.27}$$

where

$$X_1 = A_{11}A_{21} + A_{12}A_{22} + B_{11}B_{21} + B_{12}B_{22} + C_{11}C_{21} + C_{12}C_{22} + D_{11}D_{21} + D_{12}D_{22},$$

$$X_2 = A_{11}C_{21} + A_{12}C_{22} + B_{11}D_{21} + B_{12}D_{22} - C_{11}A_{21} - C_{12}A_{22} - D_{11}B_{21} - D_{12}B_{22}.$$

By Degen's eight-square identity,

$$(A_{11}^2+A_{12}^2+B_{11}^2+B_{12}^2+C_{11}^2+C_{12}^2+D_{11}^2+D_{12}^2)(A_{21}^2+A_{22}^2+B_{21}^2+B_{22}^2+C_{21}^2+C_{22}^2+D_{21}^2+D_{22}^2) = \sum_{i=1}^{8} X_i^2,$$

for X_1, X_2 as above, and

$$X_3 = A_{11}A_{22} - A_{12}A_{21} + B_{11}D_{22} - B_{12}D_{21} - C_{11}C_{22} + C_{12}C_{21} - D_{12}B_{21} + D_{11}B_{22},$$

$$X_4 = A_{11}C_{22} - A_{12}C_{21} + B_{11}B_{22} - B_{12}B_{21} - C_{12}A_{21} + C_{11}A_{22} - D_{11}D_{22} + D_{12}D_{21},$$

$$X_5 = A_{11}B_{21} - A_{12}D_{22} - B_{11}A_{21} + B_{12}C_{22} - C_{11}D_{21} - C_{12}B_{22} + D_{12}A_{22} + D_{11}C_{21},$$

$$X_6 = A_{11}D_{21} - A_{12}B_{22} - B_{11}C_{21} + B_{12}A_{22} + C_{11}B_{21} + C_{12}D_{22} - D_{11}A_{21} - D_{12}C_{22},$$

$$X_7 = A_{11}D_{22} + A_{12}B_{21} - B_{11}A_{22} - B_{12}C_{21} + C_{11}B_{22} - C_{12}D_{21} + D_{11}C_{22} - D_{12}A_{21},$$

$$X_8 = A_{11}B_{22} + A_{12}D_{21} - B_{11}C_{22} - B_{12}A_{21} - C_{11}D_{22} + C_{12}B_{21} - D_{11}A_{22} + C_{21}D_{12}.$$

Therefore (A.27) equals

$$4r(g)^2 + 2r(g)\sum_{i,j} g_{ij}^2 + \sum_{i=3}^{8} X_i^2,$$

and the proposition follows from Corollary A.11. □

Appendix B. Off-diagonal terms

In this appendix, we give a very rough estimate for the off-diagonal terms in the relative trace formula to obtain the following quantitative version of Theorem 6.3 for a fixed level N, valid when $n = 2$.

Theorem B.1. *Suppose $n = 2$, $\mathtt{k} \geq 17$, $\mathbf{S}$ is a finite set of primes, $r = \prod_{p\in\mathbf{S}} p^{r_p}$ is the similitude attached to the local test functions f_p as in* (5.1)*, and N is a fixed level prime to $\mathbf{S}$. Then with notation as in Theorem 6.3,*

$$\frac{1}{\psi(N)} \sum_{\pi\in\Pi_{\mathtt{k}}(N)} \sum_{\varphi\in E_{\mathtt{k}}(\pi,N)} \frac{c_{\sigma_1}(\varphi)\overline{c_{\sigma_2}(\varphi)}}{\|\varphi\|^2} \prod_{p\in\mathbf{S}} (\mathcal{S}f_p)(t_{\pi_p})$$

$$= \sum_{A} \int_{S_2(\mathbf{A})} f_1(\begin{pmatrix} I_2 & S \\ O & I_2 \end{pmatrix}\begin{pmatrix} A & O \\ O & r\,{}^tA^{-1} \end{pmatrix})\theta(\operatorname{tr}\sigma_1 S)dS + O\left(\frac{\mathtt{k}^{21/2}(8r)^{\mathtt{k}/2}}{N^{\mathtt{k}-12}}\right),$$

for an absolute implied constant.

Remarks: (1) The sum over A is dependent on $\mathtt{k}$, but not on N.

(2) With extra work, one can increase the power of N in the error term, and decrease the lower bound on $\mathtt{k}$.

Before specializing to the case $n=2$, we give the general form of the Fourier trace formula for a fixed level N and any n. With notation as in §5-6, define

$$I_N = \frac{1}{\psi(N)} \iint\limits_{(N(\mathbf{Q})\backslash N(\mathbf{A}))^2} K_{f_N}(n_1,n_2)\overline{\theta_1(n_1)}\theta_2(n_2)dn_1dn_2.$$

By using the spectral and geometric forms of the kernel, we have

$$\text{(B.1)} \qquad \frac{1}{\psi(N)} \sum_{\pi\in\Pi_{\mathtt{k}}(N)} \sum_{\varphi\in E_{\mathtt{k}}(\pi,N)} \frac{c_{\sigma_1}(\varphi)\overline{c_{\sigma_2}(\varphi)}}{\|\varphi\|^2} \prod_{p\in\mathbf{S}} (\mathcal{S}f_p)(t_{\pi_p}) = M(f)+E(f),$$

where

$$M(f) = \sum_A \int_{S_n(\mathbf{A})} f_1(\begin{pmatrix} I_n & S \\ O & I_n\end{pmatrix}\begin{pmatrix} A & O \\ O & r\,{}^tA^{-1}\end{pmatrix})\theta(\operatorname{tr}\sigma_1 S)dS$$

is the "diagonal" term appearing in Theorem 6.3, and

$$E(f) = \frac{1}{\psi(N)} \iint_{(N(\mathbf{Q})\backslash N(\mathbf{A}))^2} \sum_{\gamma=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in\mathbb{G}(\mathbf{Q}),C\neq O} f(n_1^{-1}\gamma n_2)\overline{\theta_1(n_1)}\theta_2(n_2)dn_1dn_2$$

is the off-diagonal contribution. (One checks readily that the integrand is indeed invariant under $N(\mathbf{Q})\times N(\mathbf{Q})$.)

It is possible to express $E(f)$ as a sum of explicit orbital integrals (cf. [KL2, §2]), with the orbits determined via the Bruhat decomposition of $\mathbb{G}(\mathbf{Q})$. For our more modest goal of proving Theorem B.1, it suffices to show that

$$\text{(B.2)} \qquad E(f) \ll \frac{\mathtt{k}^{21/2}(8r)^{\mathtt{k}/2}}{N^{\mathtt{k}-12}}$$

when $n=2$.

Lemma B.2. *For $\mathtt{k}\geq 2$, $\Delta>0$, and $a\in\mathbf{R}$,*

$$\sum_{n=-\infty}^{\infty} \frac{1}{((n+a)^2+\Delta^2)^{\mathtt{k}/2}} \leq \frac{\mathtt{k}+\Delta}{\Delta}\int_{\mathbf{R}} \frac{dx}{(x^2+\Delta^2)^{\mathtt{k}/2}}.$$

Proof. Let

$$f(x) = \frac{1}{((x+a)^2+\Delta^2)^{\mathtt{k}/2}}.$$

Then $f'(x) = -\dfrac{\mathtt{k}(x+a)}{((x+a)^2+\Delta^2)^{\mathtt{k}/2+1}}$, so that

$$|f'(x)| \leq \mathtt{k}\frac{((x+a)^2+\Delta^2)^{1/2}}{((x+a)^2+\Delta^2)^{\mathtt{k}/2+1}} = \frac{\mathtt{k}}{((x+a)^2+\Delta^2)^{1/2}}f(x) \leq \frac{\mathtt{k}}{\Delta}f(x).$$

By Euler's summation formula ([MV, p. 495]),

$$\sum_{c<n\leq d} f(n) = \int_c^d f(x)dx - f(d)\{d\} + f(c)\{c\} + \int_c^d \{x\}f'(x)dx,$$

where $\{x\} = x - [x]$ is the fractional part of x. Hence

$$\sum_{n=-\infty}^{\infty} f(n) \le \int_{\mathbf{R}} f(x)dx + \int_{\mathbf{R}} |f'(x)|dx \le (\frac{\mathtt{k}}{\Delta}+1)\int_{\mathbf{R}} f(x)dx$$
$$= \frac{\mathtt{k}+\Delta}{\Delta}\int_{\mathbf{R}} \frac{dx}{((x+a)^2+\Delta^2)^{\mathtt{k}/2}} = \frac{\mathtt{k}+\Delta}{\Delta}\int_{\mathbf{R}} \frac{dx}{(x^2+\Delta^2)^{\mathtt{k}/2}}. \qquad \square$$

Proof of Theorem B.1. As before, for any set R, we let $S_2(R)$ denote the 2×2 symmetric matrices over R. For such a matrix S, let $n_S = \left(\begin{smallmatrix} I_2 & S \\ O & I_2 \end{smallmatrix}\right)$. Because $[0,1)\times\widehat{\mathbf{Z}}$ is a fundamental domain for $\mathbf{Q}\backslash\mathbf{A}$,

$$E(f) = \frac{1}{\psi(N)} \iint_{S_2([0,1)\times\widehat{\mathbf{Z}})^2} \sum_{\gamma=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in\mathbb{G}(\mathbf{Q}), C\neq O} f(n_S^{-1}\gamma n_{S'})\theta(\operatorname{tr}\sigma_2 S' - \operatorname{tr}\sigma_1 S)\,dSdS'.$$

Assuming $n_{S_{\rm fin}}^{-1}\gamma n_{S'_{\rm fin}} \in \operatorname{supp} f_{\rm fin}$, by Proposition 5.1 we may take

$$r(n_S^{-1}\gamma n_{S'}) = r(\gamma) = \pm r, \quad (n_S^{-1}\gamma n_{S'})_{\rm fin} \in M_4(\widehat{\mathbf{Z}}), \quad C \in M_2(N\mathbf{Z}).$$

In fact, because f_∞ is supported on matrices with positive similitude, we can take $r(\gamma) = r$. By the fact that $S_{\rm fin}, S'_{\rm fin} \in S_2(\widehat{\mathbf{Z}})$, it also follows that $A, B, D \in M_2(\mathbf{Z})$.

For a square matrix g with real entries, define

$$\mathcal{Q}(g) = \sum_{i,j} g_{ij}^2$$

where g_{ij} is the (i,j)-th entry of g. Then since $|f(n_1^{-1}\gamma n_2)| \le \psi(N)|f_\infty(n_{1,\infty}^{-1}\gamma n_{2,\infty})|$, it follows by (A.26) that

$$\text{(B.3)} \quad |E(f)| \le \frac{1}{2} d_{\mathtt{k}}(8r)^{\mathtt{k}/2} \iint_{S_2([0,1))^2} \sum_{\substack{A,B,D\in\mathbf{M}_2(\mathbf{Z}),\\ O\neq C\in M_2(N\mathbf{Z})}} (2r + \mathcal{Q}(n_S^{-1}\begin{pmatrix} A & B \\ C & D\end{pmatrix} n_{S'}))^{-\mathtt{k}/2} dSdS'.$$

(The factor $1/2$ accounts for the fact that we quotient by the center of $G(\mathbf{Q})$.) We first consider the sum over B:

$$\sum_{B\in M_2(\mathbf{Z})} \left(2r + \mathcal{Q}(n_S^{-1}\begin{pmatrix} A & B \\ C & D\end{pmatrix} n_{S'})\right)^{-\mathtt{k}/2}$$
$$= \sum_{B\in M_2(\mathbf{Z})} \left(2r + \mathcal{Q}(\begin{pmatrix} A-SC & B-SD+AS'-SCS' \\ C & CS'+D\end{pmatrix})\right)^{-\mathtt{k}/2}.$$

By four applications of Lemma B.2, the above is

$$\ll \mathtt{k}^4 \int_{M_2(\mathbf{R})} \left(2r + \mathcal{Q}(\begin{pmatrix} A-SC & Y \\ C & CS'+D\end{pmatrix})\right)^{-\mathtt{k}/2} dY.$$

Summing this over A, D, we find in the same way that

$$\sum_{A,B,D\in M_2(\mathbf{Z})} \left(2r + \mathcal{Q}(n_S^{-1}\begin{pmatrix} A & B \\ C & D\end{pmatrix} n_{S'})\right)^{-\mathtt{k}/2} \ll \mathtt{k}^{12} \int_{M_2(\mathbf{R})^3} \left(2r + \mathcal{Q}(\begin{pmatrix} X & Y \\ C & Z\end{pmatrix})\right)^{-\mathtt{k}/2} dXdYdZ.$$

We now have an expression that is independent of S, S', so using (A.21), (B.3) gives

$$|E(f)| \ll \mathtt{k}^{15}(8r)^{\mathtt{k}/2} \sum_{O\neq C\in M_2(N\mathbf{Z})} \int_{M_2(\mathbf{R})^3} \left(2r + \mathcal{Q}(\begin{pmatrix} X & Y \\ C & Z\end{pmatrix})\right)^{-\mathtt{k}/2} dXdYdZ.$$

Because $C \neq O$, the above integral is bounded above by

$$\int_{M_2(\mathbf{R})^3} \mathcal{Q}(\begin{pmatrix} X & Y \\ C & Z \end{pmatrix})^{-\mathtt{k}/2} dX\,dY\,dZ = \int_{M_2(\mathbf{R})^3} (\mathcal{Q}(X)+\mathcal{Q}(Y)+\mathcal{Q}(Z)+\mathcal{Q}(C))^{-\mathtt{k}/2} dX\,dY\,dZ.$$

Replacing X, Y, Z with $\mathcal{Q}(C)^{1/2}X,\ \mathcal{Q}(C)^{1/2}Y,\ \mathcal{Q}(C)^{1/2}Z$ respectively, the above is

$$= \mathcal{Q}(C)^{-\mathtt{k}/2+6} \int_{\mathbf{R}^{12}} \frac{dX_{11}\cdots dX_{22}dY_{11}\cdots dY_{22}dZ_{11}\cdots dZ_{22}}{(1+X_{11}^2+\cdots+X_{22}^2+Y_{11}^2+\cdots+Y_{22}^2+Z_{11}^2+\cdots+Z_{22}^2)^{\mathtt{k}/2}},$$

which converges if $\mathtt{k} \geq 13$. Indeed, for such $\mathtt{k}$, we find using spherical coordinates that the above is

$$\ll \mathcal{Q}(C)^{-\mathtt{k}/2+6} \int_0^\infty \frac{\rho^{11}}{(1+\rho^2)^{\mathtt{k}/2}} d\rho \ll \mathtt{k}^{-6}(C_{11}^2+C_{12}^2+C_{21}^2+C_{22}^2)^{-\mathtt{k}/2+6}$$

since the integral over ρ is equal to $\frac{1}{2}B(6, \frac{\mathtt{k}}{2}-6)$ for the Beta function $B(x,y)$, and by Stirling's formula, $B(6, \frac{\mathtt{k}}{2}-6) \sim \Gamma(6)(\frac{\mathtt{k}}{2}-6)^{-6}$ as $\mathtt{k} \to \infty$.

Finally, we need to sum over all $O \neq C \in M_2(N\mathbf{Z})$. Write $C_{ij} = NC'_{ij}$. We may assume $C_{11} \neq 0$. (The other cases can be handled by the exactly the same method.) Thus

$$|E(f)| \ll \frac{\mathtt{k}^9 (8r)^{\mathtt{k}/2}}{N^{\mathtt{k}-12}} \sum_{C'_{ij}\in\mathbf{Z},\, C'_{11}\neq 0} ((C'_{11})^2+(C'_{12})^2+(C'_{21})^2+(C'_{22})^2)^{-\mathtt{k}/2+6}.$$

$$\ll \frac{\mathtt{k}^{12}(8r)^{\mathtt{k}/2}}{N^{\mathtt{k}-12}} \sum_{C'_{11}\neq 0} \int_{\mathbf{R}^3} ((C'_{11})^2+x^2+y^2+z^2)^{-\mathtt{k}/2+6} dx\,dy\,dz \quad \text{(by Lemma B.2)}$$

$$= \frac{\mathtt{k}^{12}(8r)^{\mathtt{k}/2}}{N^{\mathtt{k}-12}} \sum_{c\neq 0} c^{-\mathtt{k}+15} \int_{\mathbf{R}^3} (1+x^2+y^2+z^2)^{-\mathtt{k}/2+6} dx\,dy\,dz$$

$$\ll \frac{\mathtt{k}^{12}(8r)^{\mathtt{k}/2}}{N^{\mathtt{k}-12}} \sum_{c\neq 0} \frac{1}{c^{\mathtt{k}-15}} \int_0^\infty \frac{\rho^2}{(1+\rho^2)^{\mathtt{k}/2-6}} d\rho.$$

This converges as long as $\mathtt{k} \geq 17$. The integral is equal to $\frac{1}{2}B(\frac{3}{2}, \frac{\mathtt{k}-15}{2}) \sim \frac{1}{2}\Gamma(\frac{3}{2})(\frac{\mathtt{k}-15}{2})^{-3/2} \ll \mathtt{k}^{-3/2}$. Hence for $\mathtt{k} \geq 17$, the above is $\ll \frac{\mathtt{k}^{21/2}(8r)^{\mathtt{k}/2}}{N^{\mathtt{k}-12}}$ for an absolute implied constant. This proves (B.2), and Theorem B.1 follows. $\square$

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