WEIGHTED DISTRIBUTION OF LOW-LYING ZEROS OF GL(2) L-FUNCTIONS

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Abstract. We show that if the zeros of an automorphic L-function are weighted by the central value of the L-function or a quadratic imaginary base change, then for certain families of holomorphic GL(2) newforms, it has the effect of changing the distribution type of low-lying zeros from orthogonal to symplectic, for test functions whose Fourier transforms have sufficiently restricted support. However, if the L-value is twisted by a nontrivial quadratic character, the distribution type remains orthogonal. The proofs involve two vertical equidistribution results for Hecke eigenvalues weighted by central twisted L-values. One of these is an extension of a result of Ramakrishnan and Rogawski, and the other is new and involves an asymmetric probability measure that has not appeared in previous equidistribution results for GL(2).

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1. Introduction

According to the density conjecture of Katz and Sarnak, for any family of L-functions, the zeros lying close to the real axis are equidistributed according to one of a handful of possible symmetry types coming from compact classical groups ([KS1], [KS2]). More precisely, given an L-function $L(s,f)$, denote its nontrivial zeros by $\rho_f = \frac{1}{2} + i\gamma_f$, and define the 1-level density

$$\mathcal{D}(f,\phi) = \sum_{\rho_f} \phi\Big(\frac{\gamma_f \log Q_f}{2\pi}\Big),$$

where Q_f is the analytic conductor of f, and ϕ is a test function. The conjecture predicts that for any family $\mathcal{F} = \bigcup \mathcal{F}_n$ of automorphic forms, with each $\mathcal{F}_n$ finite, there exists a family G of classical compact groups (being one of O, SO(even), SO(odd), Sp, or U) such that for any even Schwartz function ϕ with compactly supported Fourier transform $\widehat{\phi}$,

$$\lim_{n\to\infty} \frac{\sum_{f\in\mathcal{F}_n} \mathcal{D}(f,\phi)}{|\mathcal{F}_n|} = \int_{-\infty}^{\infty} \phi(x) W_G(x)\,dx.$$

Here, $W_G(x)$ is the limiting distribution of the 1-level density attached to the eigenvalues of random matrices in G, as the rank tends to ∞. Of particular relevance to us here are

$$W_{\mathrm{O}}(x) = 1 + \frac{1}{2}\delta_0(x)$$

and

$$W_{\mathrm{Sp}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x},$$

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where δ_0 is the Dirac distribution at 0. As a distribution, $W_{\mathrm{Sp}}(x)$ coincides with $1-\frac{1}{2}\delta_0(x)$ when, as will always be the case for us here, $\widehat{\phi}$ is supported in $(-1,1)$. This is a consequence of the Plancherel formula ([ILS, (1.34)]).

Averages involving automorphic forms are naturally studied using the trace formula. Many variants of the trace formula involve weighting factors, such as the harmonic weight $\frac{|a_f(1)|^2}{\|f\|^2}$ that arises in the Petersson formula. In some cases, including that of GL(2) newforms, the presence of this weight is innocuous in the sense that it does not affect the distribution of low-lying zeros, [Mi]. However, in the case of zeros of GSp(4) spinor L-functions, Kowalski, Saha and Tsimmerman found that the analogous harmonic weight leads to a *symplectic* distribution, despite a heuristic suggesting that the unweighted distribution is *orthogonal*, [KST]. They gave a striking interpretation of this as evidence for Böcherer's conjecture, according to which the Fourier coefficient arising in the weight contains arithmetic information in the form of central L-values.

The question thus arises: in the simplest case of holomorphic GL(2) cusp forms, if we weight the low-lying zeros by central L-values, does it likewise change the distribution from orthogonal to symplectic? The answer depends on the type of L-function used in the weight, as we illustrate below using several families with suitably restricted test functions. We do not use the Petersson formula, but rather the relative trace formulas developed in [RR1] and [JK], in which central L-values appear directly.

In Theorem 1.1, we consider the effect of weighting by the central L-value and a Fourier coefficient. We show for two different families of holomorphic newforms that the weighted distribution of low-lying zeros is symplectic when $\widehat{\phi}$ is supported in $(-\frac{1}{2},\frac{1}{2})$. However, if the L-value is twisted by a nontrivial quadratic character, the weighted distribution orthogonal. In Theorem 1.2, we show that the zeros of L-functions attached to newforms of prime level $N\to\infty$, when weighted by an imaginary quadratic base change central L-value, have symplectic distribution for a restricted class of ϕ. We do not assume any version of the Generalized Riemann Hypothesis, though it motivates the definition of one-level density, and its use can enable one to extend the allowable range of support of $\widehat{\phi}$ ([BBDDM], [ILS]). Of course, it would be of interest to widen the range of support beyond $(-1,1)$ because the nature of the measure W_{Sp} changes there.

Theorem 1.1. *Let χ be a primitive real Dirichlet character of modulus $D\geq 1$. Let $r>0$ be an integer relatively prime to D. For a holomorphic newform f, define the weight*

$$w_f=\frac{\Lambda(\frac{1}{2},f\times\chi)\overline{a_f(r)}}{\|f\|^2} \tag{1.1}$$

for the completed L-function $\Lambda(s,f\times\chi)$ defined in (2.3) below. Let ϕ be any even Schwartz function whose Fourier transform $\widehat{\phi}(y)=\int_{-\infty}^{\infty}\phi(x)e^{-2\pi ixy}dx$ is supported inside $(-\frac{1}{2},\frac{1}{2})$. Then

$$\lim_{n\to\infty}\frac{\sum_{f\in\mathcal{F}_n}\mathcal{D}(f,\phi)w_f}{\sum_{f\in\mathcal{F}_n}w_f}=\begin{cases}\displaystyle\int_{-\infty}^{\infty}\phi(x)W_{\mathrm{Sp}}(x)dx, & \chi \text{ trivial}\\[2ex] \displaystyle\int_{-\infty}^{\infty}\phi(x)W_{\mathrm{O}}(x)dx, & \chi \text{ nontrivial}\end{cases}$$

in each of the following situations:

- $n = k$ *and* $\mathcal{F}_n = F_k(1)$ *is the set of newforms of level* 1 *with the weight* k *ranging over even numbers satisfying* $\tau(\chi)^2 \neq -i^k D$ *for the Gauss sum* $\tau(\chi) = \sum_{m=1}^{D} \chi(m) e^{2\pi i m/D}$.
- $\mathcal{F}_n = \mathcal{F}_k(N)^{new}$ *(with* $N + k \to \infty$ *as* $n \to \infty$*) is the set of newforms of prime level* $N \nmid rD$*, and even weight* $k > 2$ *chosen so that* $\tau(\chi)^2 = -i^k D$*, or equivalently,* $\chi(-1) = -i^k$.

Remarks: (1) Iwaniec, Luo and Sarnak showed that in the unweighted case, the distribution is orthogonal, [ILS].
(2) We prove Theorem 1.1 in §4. It is shown there that in the second case, if k is fixed and $N \to \infty$, the allowable support of $\widehat{\phi}$ can be widened to $[-\alpha, \alpha]$ for $\alpha < 1 - \frac{1}{k}$.
(3) The conditions involving $\tau(\chi)$ reflect the functional equation (2.4) when $N = 1$. Since $\chi = \overline{\chi}$, the condition $\frac{i^k \tau(\chi)^2}{D} = -1$ forces the L-function to vanish at $s = \frac{1}{2}$. In the first case above (where $N = 1$), the given condition keeps this from happening, and guarantees that the sum of the weights is nonzero when k is sufficiently large. In the second case where N is prime, the given condition is desirable since it causes the weights attached to the oldforms to vanish, leaving us with an expression involving only newforms.

Theorem 1.2. *Fix a quadratic discriminant* $-D < 0$*, and let* $\chi = \chi_{-D}$ *be the associated primitive quadratic Dirichlet character of conductor* D*. Let* $\mathcal{F}_N = \mathcal{F}_k(N)^{new}$ *be the set of holomorphic newforms of prime level* N *and fixed even weight* $k > 2$*. For* $f \in \mathcal{F}_N$*, define the weight*

$$w_f = \frac{\Lambda(\frac{1}{2}, f \times \chi)\Lambda(\frac{1}{2}, f)}{\|f\|^2}.$$

Then for any even Schwartz function ϕ *with* $\widehat{\phi}$ *supported inside* $(-\frac{k-2}{k}, \frac{k-2}{k})$*, we have*

$$\lim_{N\to\infty} \frac{\sum_{f\in\mathcal{F}_N} \mathcal{D}(f,\phi) w_f}{\sum_{f\in\mathcal{F}_N} w_f} = \int_{-\infty}^{\infty} \phi(x) W_{\mathrm{Sp}}(x) dx.$$

Here, N *ranges over prime values for which* $\chi(-N) = 1$.

Remark: The forms f may in fact be taken to range over the family $\mathcal{F}_N^+$ of newforms with epsilon factor $\varepsilon_f = 1$ since $\Lambda(\frac{1}{2}, f) = 0$ when $\varepsilon_f = -1$. The family $\mathcal{F}_N^+$ has symmetry type SO(even) ([ILS]).

The proof is given in §6. It involves an extension (Theorem 7.5) of the relative trace formula [RR1] of Ramakrishnan and Rogawski. The theorem of [RR1] has been extended in various other ways ([FW], [T], [Su], [SuT]), so presumably one could similarly extend scope of the above theorem with some extra work.

Theorems 1.1 and 1.2 are derived from weighted equidistribution results for Hecke eigenvalues at a fixed prime p, described in more detail below. In each case, the relevant measure is dependent on the value $\chi(p) = \pm 1$. This dependence plays an interesting role in the proof of the above theorems. From the explicit formula, we need to consider the sum over p of the weighted average of the p-th Hecke eigenvalue. Because of the nature of the relevant measure, the contribution of the primes satisfying $\chi(p) = 1$ differs from that of the primes satsifying $\chi(p) = -1$.

We then apply the prime number theorem for arithmetic progressions to get the results.

In general, the Satake parameters of holomorphic modular forms are known to satisfy many equidistribution laws. Foremost is the celebrated Sato-Tate conjecture (proven in [BLGHT]), which asserts that for a fixed non-CM cusp form $f \in S_k(N)$, the sequence of normalized Hecke eigenvalues at the unramified primes p (in their natural ordering) is equidistributed in $[-2,2]$ relative to the Sato-Tate measure

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\,dx & \text{if } -2 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

In a different direction, one can fix the prime p and allow the cusp form to vary within a family, possibly with weights. In this setting there are strikingly many different equidistribution results for GL(2) in the literature.[2] We summarize many of these in Figure 1, giving references for the precise statements in each case.

Family	Weights	Measure	References
$S_k(N)$ $N+k\to\infty$	1	Plancherel: $\frac{p+1}{(p^{1/2}+p^{-1/2})^2-x^2}\mu_\infty$	[Se], [CDF], [Li2]
$L^2_0(\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H})$ $\lambda_j\le T^2, T\to\infty$	1	Plancherel	[Sa]
$S_k(N)$ $N+k\to\infty$	$\frac{\lvert a_f(r)\rvert^2}{\lVert f\rVert^2}$	Sato-Tate (μ_∞)	[Li1], [KL2]
$L^2_0(\Gamma_0(N)\backslash\mathbf{H})$ $N\to\infty$	$\frac{\lvert a_{u_j}(r)\rvert^2 h(\lambda_j)}{\lVert u_j\rVert^2}$	Sato-Tate	[KL4]
$L^2_0(\mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H})$ $\lambda_j\le T^2, T\to\infty$	$\frac{\lvert a_{u_j}(r)\rvert^2}{\lVert u_j\rVert^2}$	Sato-Tate	[Br], [BBR], [BrM]
$S_k(N)$ $N+k\to\infty$	$\frac{\Lambda(\frac12,f\times\chi)\Lambda(\frac12,f)}{\lVert f\rVert^2}$ for χ quadratic	$\frac{L_p(\frac12,x,\chi)L_p(\frac12,x)}{L_p(1,\chi)}\mu_\infty$	[RR1], [FW], [SuT], Cor. 5.2 below, (Also [Su], [T] for Maass forms of increasing level)
$S_k(N)$ $N\to\infty$	regular matrix summation involving $\frac{1}{\lVert f\rVert^2}$	$\frac12(1-\frac{x^2}{4})^{-1}\mu_\infty$	[GMR]
$S_k(N)$ $N+k\to\infty$	$\frac{\overline{a_f(r)}\,\Lambda(s,f\times\chi)}{\lVert f\rVert^2}$	$L_p(s,x,\chi)\mu_\infty$	Theorem 3.2 below

FIGURE 1. Various fixed-p equidistribution results for Hecke eigenvalues on GL(2). (See (2.5) for the definition of $L_p(s,x,\chi)$.)

The last of these examples is new. Theorem 3.2 is a generalized and quantitative version of the following. Notation is defined precisely in Section 2.

Theorem 1.3. *Let χ be a primitive real Dirichlet character of conductor $D \ge 1$ coprime to N, let $p \nmid DN$ be a fixed prime, and let $\mathcal{F}_{N,k}$ be an orthogonal basis for the space $S_k(N)$ of cusp forms, consisting of eigenfunctions of the Hecke operator T_p, with first Fourier coefficient* 1*. Then assuming $N > 1$ and $k > 2$,*

[2] Some of these have been extended to groups of higher rank, e.g., [Z], [BBR], [ST], [MT]. There are also some hybrid results for GL(2) with both p and the conductor tending to ∞, [Na], [W].

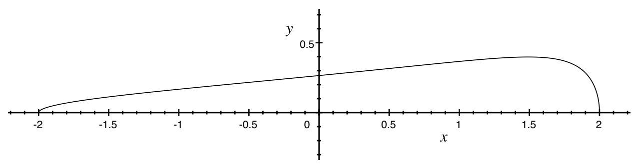
the Hecke eigenvalues $\lambda_f(p) \in [-2,2]$ *for* $f \in \mathcal{F}_{N,k}$*, when weighted by the central twisted* L*-values* $w_f = \frac{\Lambda(\frac{1}{2}, f\times\chi)}{\|f\|^2}$*, become equidistributed in* $[-2,2]$ *with respect to the probability measure*

$$d\mu_p(x) = \frac{p}{(p+1) - x\chi(p)\sqrt{p}}\, d\mu_\infty(x)$$

as $N + k \to \infty$*.*

There is a natural interpretation of the measure appearing in the above theorem. See the remark after Theorem 3.2.

Interestingly, the measure is not symmetric, though as expected it converges to the Sato-Tate measure as $p \to \infty$. It is plotted below in the case $p = 5$ when $\chi(5) = 1$:



If $\chi(p) = -1$, there is an analogous negative bias. We emphasize that χ is allowed to be trivial. In the generalized version (Theorem 3.2), χ need not be real, and we do not specialize the L-value to $s = \frac{1}{2}$.

We also give, in Corollary 5.2, another result of this nature, namely that for newforms $f \in S_k(N)$ with N prime, the $\lambda_f(p)$, when weighted as in Theorem 1.2, become equidistributed in the limit as $N + k \to \infty$ relative to the measure η_p given in the sixth row of the above table. This is essentially the main result of [RR1], which treats the case $N \to \infty$. We obtain the more general statement by keeping track of the dependence on k in their calculations. The measure η_p depends on χ. It exhibits a similar positive bias precisely when $\chi(p) = 1$. When $\chi(p) = -1$, it coincides with the Plancherel measure, which is even.

2. Preliminaries on modular forms

Fix a Dirichlet character ψ modulo N, and let $S_k(N,\psi)$ be the space of holomorphic cusp forms f on the complex upper half-plane $\mathbf{H}$ that transform under the action of $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) |\, c \in N\mathbf{Z}\}$ according to

$$f(\frac{az+b}{cz+d}) = \psi(d)(cz+d)^k f(z).$$

We normalize the Petersson inner product on $S_k(N,\psi)$ by

$$\|f\|^2 = \frac{1}{\nu(N)} \iint_{\Gamma_0(N)\backslash\mathbf{H}} |f(z)|^2 y^k \frac{dx\,dy}{y^2}, \tag{2.1}$$

where

$$\nu(N) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)].$$

For us, a *Hecke eigenform* is a simultaneous eigenfunction of the Hecke operators

$$T_n f(z) = n^{k-1} \sum_{\substack{ad=n,\\ a>0}} \sum_{b=0}^{d-1} \psi(a) d^{-k} f(\frac{az+b}{d})$$

for $(n, N) = 1$, normalized to have first Fourier coefficient 1. Given a Hecke eigenform

$$f(z) = \sum_{n>0} a_f(n) q^n \qquad (q = e^{2\pi i z}),$$

for a prime $p \nmid N$ we fix a complex square root $\psi(p)^{1/2}$ and define the normalized p-power Hecke eigenvalue

$$\lambda_f(p^\ell) = \frac{a_f(p^\ell)}{\psi(p)^{\ell/2} p^{\ell(k-1)/2}} \quad (\ell \geq 0).$$

By Deligne's theorem $\lambda_f(p) \in [-2, 2]$, and our interest is in the distribution of these numbers as f varies. For any integer $\ell \geq 0$,

$$\lambda_f(p^\ell) = X_\ell(\lambda_f(p)),$$

where X_ℓ is the Chebyshev polynomial of degree ℓ defined by $X_\ell(2\cos\theta) = \frac{\sin((\ell+1)\theta)}{\sin\theta}$ (see, e.g., [KL1, Prop. 29.8], where ω' corresponds to ψ^{-1}). Equivalently,

$$a_f(p^\ell) = \psi(p)^{\ell/2} p^{\ell(k-1)/2} X_\ell(\lambda_f(p)). \tag{2.2}$$

Fix an integer D with $(D, N) = 1$, and let χ be a primitive Dirichlet character modulo D. The χ-twisted L-function of f is given for $\mathrm{Re}(s) > 1$ by the Dirichlet series

$$L(s, f \times \chi) = \sum_{n>0} \frac{\chi(n) a_f(n)}{n^{s+\frac{k-1}{2}}}.$$

The completed L-function

$$\Lambda(s, f \times \chi) = (2\pi)^{-s-\frac{k-1}{2}} \Gamma(s + \tfrac{k-1}{2}) L(s, f \times \chi) \tag{2.3}$$

has an analytic continuation to the complex plane and satisfies a functional equation relating s to $1 - s$, which takes the form

$$\Lambda(s, f \times \chi) = \frac{i^k}{D^{2s-1}} \frac{\tau(\chi)^2}{D} \Lambda(1 - s, f \times \overline{\chi}) \tag{2.4}$$

when $N = 1$. Here, $\tau(\chi) = \sum_{m=1}^{D} \chi(m) e^{2\pi i m/D}$ is the Gauss sum attached to χ.

Given $x \in [-2, 2]$ and $p \nmid DN$, there is a unique unramified unitary representation $\pi_{x,p}$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ with Satake parameters α_p, β_p satisfying $\alpha_p + \beta_p = x\psi(p)^{1/2}$ and $\alpha_p \beta_p = \psi(p)$. We denote its twisted L-factor by

$$L_p(s, x, \chi) = (1 - x\psi(p)^{1/2} \chi(p) p^{-s} + \psi(p) \chi(p)^2 p^{-2s})^{-1}. \tag{2.5}$$

With this notation, the local L-factor of $L(s, f \times \chi)$ is

$$L_p(s, f \times \chi) = L_p(s, \lambda_f(p), \chi).$$

3. Weighted equidistribution of Hecke eigenvalues I

Fix a weight $k > 2$ and a level $N > 1$, and let

$$\mathcal{F} = \mathcal{F}_{N,k} = \mathcal{F}_k(N, \psi)$$

be an orthogonal basis for $S_k(N, \psi)$ consisting of Hecke eigenforms. Fix D and χ as above, and fix integers $p, r \in \mathbf{Z}^+$ with p prime, $p \nmid rN$ and $(r, D) = 1$. In this section, we do not assume that $\chi^2 = 1$ unless explicitly stated. For each $f \in \mathcal{F}$, define the (complex) weight

$$w_f = \frac{\overline{a_f(r)} \Lambda(s, f \times \chi)}{\|f\|^2}. \tag{3.1}$$

Then for all $s=\sigma+i\tau$ in the strip $1-\frac{k-1}{2}<\sigma<\frac{k-1}{2}$, by Theorem 1.1 of [JK] (which is a twisted version of the main theorem of [KL3]), we have

$$\begin{aligned}\frac{1}{\nu(N)}\sum_{f\in\mathcal{F}} w_f a_f(n) =&\frac{2^{k-1}(2\pi rn)^{\frac{k-1}{2}-s}}{(k-2)!}\Gamma(s+\tfrac{k-1}{2})\psi(n)\chi(rn)\\ &+O\left(\frac{(4\pi rn)^{k-1}D^{\frac{k}{2}-\sigma}\varphi(D)}{N^{\sigma+\frac{k-1}{2}}(k-2)!}\right)\end{aligned}$$

for all integers n relatively prime to rDN. (We have adjusted for the fact that in [JK] the L-function is normalized to have central point $\frac{k}{2}$, whereas here the central point is $\frac12$.) The implied constant is explicit in [JK], and depends only on $\mathrm{Im}(s)$. Taking $n=p^\ell$ and substituting (2.2), the above becomes

$$\frac{1}{\nu(N)}\sum_{f\in\mathcal{F}} w_f X_\ell(\lambda_f(p)) = F_\ell+E_\ell, \tag{3.2}$$

where

$$F_\ell=\left(\psi(p)^{1/2}\chi(p)p^{-s}\right)^\ell\ \frac{2^{k-1}(2\pi r)^{\frac{k-1}{2}-s}\chi(r)}{(k-2)!}\Gamma(s+\tfrac{k-1}{2}), \tag{3.3}$$

and E_ℓ is an error term satisfying

$$E_\ell \ll p^{\frac{\ell(k-1)}{2}}\frac{(4\pi r)^{k-1}D^{\frac{k}{2}-\sigma}\varphi(D)}{N^{\sigma+\frac{k-1}{2}}(k-2)!}. \tag{3.4}$$

Proposition 3.1. *For any $\ell\ge 0$ and $0<\sigma<1$,*

$$\frac{\sum_{f\in\mathcal{F}_k(N,\psi)} w_f X_\ell(\lambda_f(p))}{\sum_{f\in\mathcal{F}_k(N,\psi)} w_f} = \left[\psi(p)^{1/2}\chi(p)p^{-s}\right]^\ell + O\left(\frac{p^{\frac{\ell(k-1)}{2}}(4\pi rDe)^{k/2}}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\right), \tag{3.5}$$

where the implied constant depends only on r,s,D.

Remark: When $N>1$, it is shown in [JK, §9] that the sum of the weights is nonzero when $N+k$ is sufficiently large. When $N=1$, this is can only be verified under certain extra conditions mentioned in Theorem 3.2 below.

Proof. In the notation of (3.2), the left-hand side of (3.5) is

$$\frac{F_\ell+E_\ell}{F_0+E_0}=\frac{F_\ell}{F_0}+\frac{E_\ell-\frac{F_\ell}{F_0}E_0}{F_0+E_0}.$$

This will immediately give (3.5) once we show that the second term on the right-hand side has the desired rate of decay. If we denote the right-hand side of (3.4) by $p^{\frac{\ell(k-1)}{2}}C_0$, then

$$\begin{aligned}\frac{E_\ell-\frac{F_\ell}{F_0}E_0}{F_0+E_0}&=\frac{E_\ell-\psi(p)^{\ell/2}p^{-\ell s}E_0}{F_0+E_0}\ll\frac{(p^{\frac{\ell(k-1)}{2}}+p^{-\ell\sigma})C_0}{F_0+E_0}\\ &\ll\frac{p^{\frac{\ell(k-1)}{2}}C_0}{F_0+E_0}=p^{\frac{\ell(k-1)}{2}}\frac{\frac{C_0}{F_0}}{1+\frac{E_0}{F_0}}.\end{aligned}$$

In §9 of [JK] (taking $n=1$), it is shown that

$$\frac{E_0}{F_0}\ll\frac{C_0}{F_0}\ll\frac{(4\pi rDe)^{k/2}}{N^{(k-1)/2}k^{k/2-1}},$$

where the implied constant depends on r, s, D. The proposition follows. $\square$

Define a measure

$$d\mu_{p,s,\chi}(x) = \sum_{\ell=0}^{\infty} \left[\psi(p)^{1/2}\chi(p)p^{-s}\right]^{\ell} X_\ell(x)\, d\mu_\infty(x), \tag{3.6}$$

where as before, μ_∞ is the Sato-Tate measure on $\mathbf{R}$ with support $[-2,2]$, and X_ℓ is the Chebyshev polynomial. The infinite series is absolutely convergent provided $|x| \le 2$ and $\mathrm{Re}(s) > 0$. Indeed, if $|x| \le 2$ and $|t| < 1$, we have the well-known identity

$$\sum_{\ell=0}^{\infty} t^\ell X_\ell(x) = \frac{1}{1 - xt + t^2}. \tag{3.7}$$

Therefore

$$\begin{aligned} d\mu_{p,s,\chi}(x) &= \frac{1}{1 - x\psi(p)^{1/2}\chi(p)p^{-s} + \psi(p)\chi(p)^2p^{-2s}} d\mu_\infty(x) \\ &= L_p(s,x,\chi) d\mu_\infty(x) \end{aligned} \tag{3.8}$$

in the notation of (2.5). This is a complex-valued probability measure since, by (3.6) and the orthonormality of the $X_\ell(x)$ relative to μ_∞, $\int X_0(x) d\mu_{p,s,\chi} = 1$. We note that when $s = \frac{1}{2}$, ψ is trivial, and χ is real,

$$d\mu_{p,\frac{1}{2},\chi} = \frac{p}{(p+1) - x\chi(p)\sqrt{p}}\, d\mu_\infty(x)$$

is the measure given in Theorem 1.3.

Theorem 3.2. *Fix s in the critical strip $0 < \mathrm{Re}(s) < 1$, let $N > 1$ be coprime to rD, let $k > 2$, let ψ be a Dirichlet character whose conductor divides N, and fix a prime $p \nmid rND$. Define weights w_f as in* (3.1)*. Then the Hecke eigenvalues $\lambda_f(p)$ for $f \in \mathcal{F}_k(N,\psi)$ become w_f-equidistributed in $[-2,2]$ relative to the measure $\mu_{p,s,\chi}$ as $N + k \to \infty$. In other words, for any continuous function ϕ on $\mathbf{R}$,*

$$\lim_{N+k\to\infty} \frac{\sum_{f\in\mathcal{F}_k(N,\psi)} w_f \phi(\lambda_f(p))}{\sum_{f\in\mathcal{F}_k(N,\psi)} w_f} = \int_{\mathbf{R}} \phi\, d\mu_{p,s,\chi}. \tag{3.9}$$

Moreover, if ϕ is a polynomial of degree d, then

$$\frac{\sum_{f\in\mathcal{F}_k(N,\psi)} w_f \phi(\lambda_f(p))}{\sum_{f\in\mathcal{F}_k(N,\psi)} w_f} = \int_{\mathbf{R}} \phi\, d\mu_{p,s,\chi} + O\left(\frac{p^{\frac{d(k-1)}{2}}(4\pi rDe)^{k/2}}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\|\phi\|_{ST}\right), \tag{3.10}$$

where $\|\phi\|_{ST}$ is the norm of ϕ in $L^2(\mathbf{R},\mu_\infty)$.

When $N = 1$, the equidistribution assertion (3.9) *still holds, provided $\chi^2 = 1$, $s = \frac{1}{2}$ and $\frac{i^k\tau(\chi)^2}{D} \neq -1$.*

Remark: The measure $\mu_{p,s,\chi}$ appearing here is natural for the following reason. The weight w_f depends directly on $\lambda_f(p)$ via the local L-factor $L_p(s, f\times\chi) = L_p(s,\lambda_f(p),\chi)$ (in the notation of (2.5)). Assuming the remaining L-factors do not affect the distribution of the $\lambda_f(p)$, on the left-hand side we have something resembling a Petersson-weighted average of the function $L_p(s,x,\chi)\phi(x)$ at the points $\lambda_f(p)$, which, in view of the equidistribution result [Li1], tends to the integral of this function against the Sato-Tate measure. This is exactly what appears on the right-hand side of (3.9).

Proof. First take $N > 1$. By the fact that the Chebyshev polynomials are orthonormal relative to the Sato-Tate measure μ_∞, we see from (3.6) that (3.5) gives (3.9) with $\phi = X_\ell$ for $\ell \geq 0$. By linearity it holds if ϕ is any polynomial, so by Weierstrass approximation, (3.9) holds for all continuous functions.

Since $\|X_\ell\|_{ST} = 1$ for all ℓ, Proposition 3.1 gives (3.10) when $\phi = X_\ell$. For an arbitrary polynomial ϕ of degree d, we may write $\phi = \sum_{\ell=0}^{d} \langle \phi, X_\ell \rangle X_\ell$, so denoting the left-hand side of (3.10) by $\mathcal{E}(\phi)$, we have

$$\left| \mathcal{E}(\phi) - \int \phi \, d\mu_{p,s,\chi} \right| = \left| \sum_{\ell=0}^{d} \langle \phi, X_\ell \rangle \left(\mathcal{E}(X_\ell) - \int X_\ell \, d\mu_{p,s,\chi} \right) \right|$$

Applying (3.5) and the Schwarz inequality $|\langle \phi, X_\ell \rangle| \leq \|\phi\|_{ST}$, the above is

$$\ll \|\phi\|_{ST} \frac{(4\pi r D e)^{k/2}}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \sum_{\ell=0}^{d} p^{\frac{\ell(k-1)}{2}},$$

and (3.10) follows.

Now suppose $N = 1$, $\chi^2 = 1$, and $s = \frac{1}{2}$. Then there is an extra main term in [JK, Theorem 1.1], so that in place of (3.3), we have

$$F_\ell = (\chi(p)p^{-1/2})^\ell \, \frac{2^{k-1}(2\pi r)^{\frac{k}{2}-1}\chi(r)}{(k-2)!} \Gamma(\tfrac{k}{2}) \left[1 + i^k \frac{\tau(\chi)^2}{D} \right].$$

(The extra main term contains the factor $\overline{\chi(p^\ell r)}$, so we we have imposed $\chi^2 = 1$ to make this equal to $\chi(p)^\ell \chi(r)$.) The rest of the argument then goes through as above, provided the bracketed expression is nonzero. □

4. Low-lying zeros I

In this section we derive Theorem 1.1 from the results of the previous section by standard methods (see, for example, [Ko, §9]). We will use Proposition 3.1, together with the following consequence of the explicit formula for the L-function of a holomorphic newform $f \in \mathcal{F}_k(N)^{new}$ with analytic conductor Q_f:

$$\begin{aligned} \mathcal{D}(f,\phi) =& \widehat{\phi}(0) + \frac{1}{2}\phi(0) - 2\sum_{p\nmid N} \frac{\lambda_f(p)\log p}{p^{1/2}\log Q_f} \widehat{\phi}\Big(\frac{\log p}{\log Q_f}\Big) \\ & - 2\sum_{p\nmid N} \frac{\lambda_f(p^2)\log p}{p\log Q_f} \widehat{\phi}\Big(\frac{2\log p}{\log Q_f}\Big) + O\Big(\frac{\log\log 3N}{\log Q_f}\Big). \end{aligned} \tag{4.1}$$

This holds for any even Schwartz function ϕ on $\mathbf{R}$ whose Fourier transform has compact support, [ILS, Lemma 4.1].

For the remainder of this section, χ is a real Dirichlet character, and $\mathcal{F}$ denotes one of the following families given in Theorem 1.1:

(1) $\mathcal{F} = \mathcal{F}_k(1)$, the set of Hecke eigenforms of level $N = 1$ and even weight k chosen so that $\frac{i^k\tau(\chi)^2}{D} \neq -1$.
(2) $\mathcal{F} = \mathcal{F}_k(N)^{new}$, where $N \nmid rD$ is prime. In this case, the even weight $k \geq 4$ is chosen so that $\frac{i^k\tau(\chi)^2}{D} = -1$.

We need to consider the weighted average of $\mathcal{D}(f,\phi)$ over $\mathcal{F}$. To simplify notation, given a function A on $\mathcal{F}$, we define the w-weighted average of A by

$$\mathcal{E}^w_{\mathcal{F}}(A) = \frac{\sum_{f\in\mathcal{F}} A_f w_f}{\sum_{f\in\mathcal{F}} w_f},$$

where w_f is the weight defined in (3.1) taking $s=\frac{1}{2}$. When $N=1$ and $\frac{i^k\tau(\chi)^2}{D}=-1$, or equivalently, $\chi(-1)=-i^k$, the functional equation (2.4) forces $\Lambda(\frac{1}{2}, f\times\chi)=0$ since χ is real. Hence when N is prime, the conditions imposed on k and χ ensure that $w_f=0$ for all Hecke eigenforms f of level 1 and weight k. If we set $f_N(z)=f(Nz)$ for such f, we have $\Lambda(\frac{1}{2}, f_N\times\chi)=\frac{\chi(N)}{N^{k/2}}\Lambda(\frac{1}{2}, f\times\chi)=0$ as well, so that $w_h=0$ for all h in the span of $\{f, f_N\}$. Therefore

$$\mathcal{E}^w_{\mathcal{F}}(A) = \mathcal{E}^w_{\mathcal{F}_k(N)}(A) \tag{4.2}$$

in this case, i.e., the value is unaffected if we average over an orthogonal basis for the full space $S_k(N)$, rather than restricting to newforms.

Since $Q_f = k^2N$ is constant across $\mathcal{F}$, we denote it by Q in what follows. By (4.1), we have

$$\frac{\sum_{f\in\mathcal{F}}\mathcal{D}(f,\phi)w_f}{\sum_{f\in\mathcal{F}} w_f} = \widehat{\phi}(0)+\frac{1}{2}\phi(0)+O\Big(\frac{\log\log 3N}{\log Q}\Big)$$

$$-2\sum_{p\nmid N}\frac{\mathcal{E}^w_{\mathcal{F}}(\lambda.(p))\log p}{p^{1/2}\log Q}\widehat{\phi}\Big(\frac{\log p}{\log Q}\Big) \tag{4.3}$$

$$-2\sum_{p\nmid N}\frac{\mathcal{E}^w_{\mathcal{F}}(\lambda.(p^2))\log p}{p\log Q}\widehat{\phi}\Big(\frac{2\log p}{\log Q}\Big). \tag{4.4}$$

Taking $s=\frac{1}{2}$, ψ trivial, and $\ell=1,2$ in (3.5), we have (using (4.2) when N is prime)

$$\mathcal{E}^w_{\mathcal{F}}(\lambda.(p)) = \chi(p)p^{-1/2}+O\Big(\frac{p^{\frac{k-1}{2}}R^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\Big), \tag{4.5}$$

and

$$\mathcal{E}^w_{\mathcal{F}}(\lambda.(p^2)) = \chi(p)^2p^{-1}+O\Big(\frac{p^{k-1}R^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\Big) \tag{4.6}$$

for a positive constant R depending on D and r. It is a consequence of the prime number theorem that for any real number $m>-1$,

$$\sum_{p\le x} p^m\log p \sim \frac{x^{m+1}}{m+1}$$

as $x\to\infty$. If the support of $\widehat{\phi}$ is contained in $[-\alpha,\alpha]$, the sum in (4.3) is restricted to $p\le Q^\alpha$. Therefore, the contribution to (4.3) of the error term in (4.5) is

$$\ll \frac{R^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\sum_{p\le Q^\alpha} p^{\frac{k}{2}-1}\log p \ll \frac{Q^{\frac{\alpha k}{2}}R^k}{(\frac{k}{2})N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}} = \frac{2N^{\frac{\alpha k}{2}}k^{\alpha k}R^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}}}$$

$$= O\Big(\frac{1}{\log Q}\Big),$$

provided $\alpha < \frac{1}{2}$. (If k is fixed, we only need $\alpha < 1 - \frac{1}{k}$.) The contribution to (4.4) of the error term in (4.6) is

$$\ll \frac{R^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \sum_{p \le Q^{\alpha/2}} p^{k-2} \log p \ll \frac{Q^{\frac{\alpha(k-1)}{2}} R^k}{(k-1) N^{\frac{k-1}{2}} k^{\frac{k}{2}-1}} \ll \frac{N^{\frac{\alpha(k-1)}{2}} k^{\alpha(k-1)} R^k}{N^{\frac{k-1}{2}} k^{\frac{k}{2}}},$$

which may likewise be absorbed into the error term above (4.3) if $\alpha < \frac{1}{2}$.

It remains to treat the contribution of the main terms of (4.5) and (4.6) to (4.3) and (4.4) respectively. If χ is trivial, the former yields

$$-2 \sum_{p \nmid N} \frac{\log p}{p \log Q} \widehat{\phi}\Big(\frac{\log p}{\log Q}\Big) = -2 \sum_{p} \frac{\log p}{p \log Q} \widehat{\phi}\Big(\frac{\log p}{\log Q}\Big) + O\Big(\frac{\log \log 3N}{\log Q}\Big)$$

(by (4.19$'$) of [ILS]), which in turn is

$$= -\phi(0) + O\Big(\frac{\log \log 3N}{\log Q}\Big)$$

by the prime number theorem, using the fact that ϕ is even.

On the other hand, if χ is nontrivial, then the main term of (4.5) contributes

$$-2 \sum_{p : \chi(p) = 1} \frac{\log p}{p \log Q} \widehat{\phi}\Big(\frac{\log p}{\log Q}\Big) + 2 \sum_{p : \chi(p) = -1} \frac{\log p}{p \log Q} \widehat{\phi}\Big(\frac{\log p}{\log Q}\Big) + O\Big(\frac{\log \log 3N}{\log Q}\Big).$$

The value of χ is 1 on exactly half of the primes. By the prime number theorem for arithmetic progressions, the above is

$$= -\frac{1}{2}\phi(0) + \frac{1}{2}\phi(0) + O\Big(\frac{\log \log 3N}{\log Q}\Big) = O\Big(\frac{\log \log 3N}{\log Q}\Big).$$

Lastly, for any χ, the contribution of the main term of (4.6) is

$$\ll 2 \sum_{p \nmid N} \frac{\log p}{p^2 \log Q} \Big|\widehat{\phi}\Big(\frac{2 \log p}{\log Q}\Big)\Big| = O\Big(\frac{1}{\log Q}\Big).$$

Putting everything together, we conclude that when $\alpha < \frac{1}{2}$,

$$\frac{\sum_{f \in \mathcal{F}} \mathcal{D}(f, \phi) w_f}{\sum_{f \in \mathcal{F}} w_f} = \begin{cases} \widehat{\phi}(0) - \frac{1}{2}\phi(0) + O\Big(\frac{\log \log 3N}{\log(k^2 N)}\Big), & \chi \text{ trivial} \\ \widehat{\phi}(0) + \frac{1}{2}\phi(0) + O\Big(\frac{\log \log 3N}{\log(k^2 N)}\Big), & \chi \text{ nontrivial.} \end{cases} \tag{4.7}$$

which proves Theorem 1.1.

5. Weighted equidistribution of Hecke eigenvalues II

We recall the setup from Theorem 1.2: N is prime, $\chi = \chi_{-D}$ is a primitive quadratic character with $\chi(-N) = 1$. For $k > 2$ even, we let $\mathcal{F} = \mathcal{F}_{N,k}^{new}$ be the set of holomorphic newforms of weight k and level N. For $f \in \mathcal{F}$, we define the weight

$$w_f = \frac{\Lambda(\frac{1}{2}, f \times \chi) \Lambda(\frac{1}{2}, f)}{\|f\|^2}.$$

Proposition 5.1. *With hypotheses as above, for any $\ell \ge 0$, and any prime $p \nmid ND$,*

$$\mathcal{E}_{\mathcal{F}}^{w}(\lambda.(p^\ell)) := \frac{\sum_{f \in \mathcal{F}} w_f X_\ell(\lambda_f(p))}{\sum_{f \in \mathcal{F}} w_f} = \int_{\mathbf{R}} X_\ell \, d\eta_p + O\Big(\frac{p^{\ell(\frac{k+1}{2} + \varepsilon)} D^{k/2}}{k^{1/2} N^{k/2 - \varepsilon}}\Big), \tag{5.1}$$

where

$$\eta_p(x) = \begin{cases} \dfrac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2}\mu_\infty(x) & \text{if } \chi(p) = -1 \\ \dfrac{p-1}{(p^{1/2}+p^{-1/2}-x)^2}\mu_\infty(x) & \text{if } \chi(p) = 1, \end{cases}$$

and the implied constant depends only on χ, ℓ and D,

This is essentially the main result of [RR1], but we have divided by the sum of the weights, and shown the dependence on p and k explicitly in the error term. The proof is somewhat involved, so we defer it to Section 7.

Corollary 5.2. *Assume the hypotheses above, and that $N > p^\ell D$. Then the multiset $\{\lambda_f(p) | f \in \mathcal{F}^{new}_{N,k}\}$ of normalized Hecke eigenvalues at p, when weighted as above, becomes equidistributed in $[-2, 2]$ with respect to the measure η_p as $N + k \to \infty$. Thus, for any continuous function ϕ,*

$$\lim_{N+k\to\infty} \frac{\sum_{f\in\mathcal{F}^{new}_{N,k}} w_f \phi(\lambda_f(p))}{\sum_{f\in\mathcal{F}^{new}_{N,k}} w_f} = \int \phi \, d\eta_p.$$

Moreover, if ϕ is a polynomial of degree d, then

$$\frac{\sum_{f\in\mathcal{F}} w_f \phi(\lambda_f(p))}{\sum_{f\in\mathcal{F}} w_f} = \int_{\mathbf{R}} \phi \, d\eta_p + O\Big(\|\phi\|_{ST} \frac{p^{d(\frac{k+1}{2}+\varepsilon)} D^{k/2}}{k^{1/2} N^{k/2-\varepsilon}}\Big). \tag{5.2}$$

Remarks: (1) In Theorem A of [RR1], a much stronger claim is made, namely that by Weierstrass approximation, we can take ϕ to be the characteristic function of any subinterval of $[-2, 2]$ in (5.2), preserving the error term $O(N^{-k/2+\varepsilon})$. However, because the error in (5.2) depends in a crucial way on the approximating polynomial ϕ, their argument is incomplete. Possibly one could use the method of Murty and Sinha [MS], but we have not investigated this.
(2) It is not hard to show that $\eta_p(x) = \frac{L_p(\frac12,x,\chi)L_p(\frac12,x)}{L_p(1,\chi)}\mu_\infty(x)$. So the above result may be interpreted in a manner analogous to the remark after Theorem 3.2.

Proof. Corollary 5.2 is proven in just the same way as Theorem 3.2. The hypothesis $N > p^\ell D$ ensures that the error term in (5.1) goes to 0 as $k \to \infty$. □

6. Low-lying zeros II

Here we will use Proposition 5.1 to prove Theorem 1.2. First we need to compute the integrals of the Chebyshev polynomials against the measure η_p defined in Proposition 5.1.

Proposition 6.1. *Let $r \geq 0$ be an integer. Then if $\chi(p) = -1$,*

$$\int_{-\infty}^{\infty} X_r \, d\eta_p = \begin{cases} p^{-r/2} & \text{if } r \text{ is even}, \\ 0 & \text{if } r \text{ is odd}. \end{cases}$$

If $\chi(p) = 1$, then

$$\int_{-\infty}^{\infty} X_r \, d\eta_p = (r+1)p^{-r/2}.$$

Proof. The first assertion is well-known ([Se]). For the second, using (3.7) we have

$$\frac{p-1}{(p^{1/2}+p^{-1/2}-x)^2} = \frac{1-\frac{1}{p}}{(1-p^{-1/2}x+p^{-1})^2} = (1-\tfrac{1}{p})\left[\sum_{n=0}^{\infty} p^{-n/2}X_n(x)\right]^2$$

$$= (1-\tfrac{1}{p})\left[\sum_{j=0}^{\infty} X_j(x)^2 p^{-j} + 2\sum_{m=1}^{\infty}\sum_{n=0}^{m-1} X_m(x)X_n(x)p^{-(m+n)/2}\right]. \tag{6.1}$$

By the Clebsch-Gordon formula (or by induction using $X_{n+1}(x) = xX_n(x) - X_{n-1}(x)$), we have

$$X_m(x)X_n(x) = \sum_{k=0}^{n} X_{m-n+2k}(x), \qquad (n \le m).$$

So (6.1) becomes

$$(1-\tfrac{1}{p})\left[\sum_{j=0}^{\infty}\sum_{t=0}^{j} X_{2t}(x)p^{-j} + 2\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\sum_{k=0}^{n} X_{m-n+2k}(x)p^{-(m+n)/2}\right]. \tag{6.2}$$

For the double sum,

$$\sum_{j=0}^{\infty}\sum_{t=0}^{j} X_{2t}(x)p^{-j} = \sum_{t=0}^{\infty} X_{2t}(x)\sum_{j=0}^{\infty} p^{-(j+t)} = (1-\tfrac{1}{p})^{-1}\sum_{t=0}^{\infty} X_{2t}(x)p^{-t}. \tag{6.3}$$

For the triple sum, we observe that the map $(m,n,k) \mapsto (m-n+2k, m-n, m)$ defines a bijection from

$$\{(m,n,k) |\, m \ge 1,\, 0 \le n \le m-1,\, 0 \le k \le n\}$$

to

$$\{(u,b,m) |\, u \ge 1,\, 1 \le b \le u,\, b \equiv u \bmod 2,\, m \ge \tfrac{u+b}{2}\}$$

with inverse $(u,b,m) \mapsto (m, m-b, \frac{u-b}{2})$. Therefore,

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=0}^{m-1}\sum_{k=0}^{n} X_{m-n+2k}(x)p^{-(m+n)/2} &= \sum_{u=1}^{\infty} X_u(x) \sum_{\substack{b\equiv u \bmod 2\\ 1\le b\le u}} \sum_{m=\frac{u+b}{2}}^{\infty} p^{-(2m-b)/2} \\
&= \sum_{u=1}^{\infty} X_u(x) \sum_{\substack{b\equiv u \bmod 2\\ 1\le b\le u}} p^{b/2}p^{-(u+b)/2}(1-\tfrac{1}{p})^{-1} \\
&= (1-\tfrac{1}{p})^{-1}\sum_{u=1}^{\infty} X_u(x)p^{-u/2} \sum_{\substack{b\equiv u \bmod 2\\ 1\le b\le u}} 1
\end{aligned}$$

The sum over b has the value $\frac{u}{2}$ if u is even, and $\frac{u+1}{2}$ if u is odd. Using this and (6.3), (6.2) becomes

$$\sum_{r\ge 0 \text{ even}} X_r(x)p^{-r/2} + 2\sum_{r\ge 2 \text{ even}} \tfrac{r}{2}X_r(x)p^{-r/2} + 2\sum_{r\ge 1 \text{ odd}} \tfrac{r+1}{2}X_r(u)p^{-r/2}.$$

In the middle sum, we can actually take $r \geq 0$ because of the $\frac{r}{2}$ coefficient. This proves that

$$d\eta_p(x) = \sum_{r=0}^{\infty} (r+1) p^{-r/2} X_r(x) d\mu_\infty(x). \tag{6.4}$$

The proposition now follows immediately using the orthonormality of the Chebyshev polynomials relative to $d\mu_\infty$. □

With this proposition in hand, we obtain the following two special cases of Proposition 5.1.

Corollary 6.2. *In the notation of Proposition 5.1, for any $\varepsilon > 0$,*

$$\mathcal{E}^w_{\mathcal{F}}(\lambda.(p)) = \begin{cases} 2p^{-1/2} + O(\frac{p^{\frac{k+1}{2}+\varepsilon}}{N^{k/2-\varepsilon}}) & \text{if } \chi(p) = 1 \\ \\ O(\frac{p^{\frac{k+1}{2}+\varepsilon}}{N^{k/2-\varepsilon}}) & \text{if } \chi(p) = -1, \end{cases} \tag{6.5}$$

and

$$\mathcal{E}^w_{\mathcal{F}}(\lambda.(p^2)) = \begin{cases} 3p^{-1} + O(\frac{p^{k+1+\varepsilon}}{N^{k/2-\varepsilon}}) & \text{if } \chi(p) = 1 \\ \\ p^{-1} + O(\frac{p^{k+1+\varepsilon}}{N^{k/2-\varepsilon}}) & \text{if } \chi(p) = -1, \end{cases} \tag{6.6}$$

where the implied constants depend on k and D.

We can now prove Theorem 1.2 following the method in Section 4. The contribution of the leading term of $\mathcal{E}^w_{\mathcal{F}}(\lambda.(p))$ to (4.3) is

$$-2 \sum_{\substack{p \nmid N \\ \chi(p)=1}} \frac{2\log p}{p \log Q} \widehat{\phi}(\frac{\log p}{\log Q}) = -4 \sum_{\substack{p \\ \chi(p)=1}} \frac{\log p}{p\log Q} \widehat{\phi}\Big(\frac{\log p}{\log Q}\Big) + O\Big(\frac{\log\log 3N}{\log Q}\Big)$$

Because χ is a nontrivial quadratic character, its value is 1 on exactly half of the primes. By the prime number theorem for arithmetic progressions, the above is

$$= -\phi(0) + O\Big(\frac{\log\log 3N}{\log Q}\Big).$$

As before, the contribution of the main term of $\mathcal{E}^w_{\mathcal{F}}(\lambda.(p^2))$ to (4.4) can be absorbed into the error term.

On the other hand, if $\widehat{\phi}$ is supported in $(-\alpha, \alpha)$, then inserting the error term of (6.5) into (4.3) gives an expression which is

$$\ll_k \frac{1}{N^{k/2-\varepsilon}} \sum_{p \leq Q^\alpha} p^{\frac{k+1}{2}+\varepsilon-\frac{1}{2}} \log p \ll_k \frac{N^{\alpha(k/2+1+\varepsilon)}}{N^{k/2-\varepsilon}} \ll_k \frac{1}{\log N} \tag{6.7}$$

as long as $\alpha(\frac{k}{2}+1+\varepsilon) < \frac{k}{2} - \varepsilon$. Noting that

$$1 - \frac{\frac{k}{2}-\varepsilon}{\frac{k}{2}+1+\varepsilon} = \frac{1+2\varepsilon}{\frac{k}{2}+1+\varepsilon} \leq \frac{1+\varepsilon}{k/2},$$

we see that this condition holds whenever $\alpha < 1 - \frac{2+\varepsilon}{k}$. Since we have the freedom to choose $\varepsilon > 0$ as small as we like, and k is fixed, it suffices to take $\alpha < 1 - \frac{2}{k}$. The contribution of the error term of (6.6) is likewise negligible under the same condition.

It now follows that if $\alpha < \frac{k-2}{k}$,

$$\frac{\sum_{f\in\mathcal{F}} \mathcal{D}(f,\phi)w_f}{\sum_{f\in\mathcal{F}} w_f} = \widehat{\phi}(0) - \frac{1}{2}\phi(0) + O\Big(\frac{\log\log 3N}{\log N}\Big)$$

for an implied constant depending on k. This proves Theorem 1.2.

We remark that because $Q^{\alpha k/2} = N^{\alpha k/2}k^{\alpha k}$ appears in (6.7), there is not enough decay in the k aspect in Proposition 5.1 to cancel the growth as $k \to \infty$ for any $\alpha > 0$. Therefore we cannot obtain the analog of Theorem 1.2 for the case of fixed N and $k \to \infty$ by this method.

7. The theorem of Ramakrishnan and Rogawski

The main result of [RR1] is an explicit relative trace formula on GL(2) given in §5 of their paper. On the spectral side, the term $f_p^\wedge(a_p(\varphi_j))$ appears, where $a_p(\varphi_j)$ coincides with our $\lambda_f(p)$ and $f_p^\wedge$ is the Satake transform of the local test function f_p. For our purpose, we need to choose the particular f_p whose Satake transform is equal to the Chebyshev polynomial X_ℓ. This function is given as follows. For $K_p = \mathrm{GL}_2(\mathbf{Z}_p)$ and Z_p the center of $\mathrm{GL}_2(\mathbf{Q}_p)$, let

$$M(p^\ell) = \bigcup_{\substack{i+j=\ell\\ i\geq j\geq 0}} Z_pK_p\begin{pmatrix} p^i & \\ & p^j\end{pmatrix}K_p = \bigcup_{\substack{i+j=\ell\\ i\geq j\geq 0}} Z_pK_p\begin{pmatrix} p^{i-j} & \\ & 1\end{pmatrix}K_p$$

$$= \bigcup_{j=0}^{\lfloor \frac{\ell}{2}\rfloor} Z_pK_p\begin{pmatrix} p^{\ell-2j} & \\ & 1\end{pmatrix}K_p. \tag{7.1}$$

Define $f_p : \mathrm{GL}_2(\mathbf{Q}_p) \longrightarrow \mathbf{C}$ by

$$f_p(g) = \begin{cases} p^{-\ell/2} & \text{if } g \in M(p^\ell)\\ 0 & \text{otherwise.}\end{cases} \tag{7.2}$$

Proposition 7.1. *For f_p as above, and any newform $f \in S_k(N)$,*

$$f_p^\wedge(\lambda_f(p)) = X_\ell(\lambda_f(p)).$$

Proof. Let π_p be the unramified principal series representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ determined by the cusp form f. Denoting its Satake parameters by $\{\alpha, \alpha^{-1}\}$, we have $\alpha + \alpha^{-1} = \lambda_f(p)$. By definition, $f_p^\wedge(\lambda_f(p))$ is the eigenvalue of $\pi_p(f_p)$ acting on the unique K_p-fixed vector of π_p. When f_p is the characteristic function of $M(p^\ell)$, it is shown in [KL2, Prop. 4.5] that, in our current notation, $p^{-\ell/2}f_p^\wedge(\lambda_f(p)) = X_\ell(\lambda_f(p))$. Therefore, upon scaling the characteristic function by $p^{-\ell/2}$ we get the desired result. □

With this choice, the spectral side of the relative trace formula in [RR1, Prop. 4.1] becomes

$$\sum_{f\in\mathcal{F}_{N,k}^{new}} w_f X_\ell(\lambda_f(p)), \tag{7.3}$$

where $w_f = \frac{\Lambda(\frac{1}{2},f\times\chi)\Lambda(\frac{1}{2},f)}{\|f\|^2}$, N is prime, $\chi(-N) = 1$, and $k > 2$ is even. The geometric side has a main term, namely

$$2c_k\nu(N)L(1\,\chi)\int_{-\infty}^{\infty} X_\ell\, d\eta_p,$$

where $c_k = 2^k \frac{(k-1)}{4\pi} B(\frac{k}{2}, \frac{k}{2})$, [RR2]. The remaining contribution of the geometric side is the sum of the so-called regular terms:

$$I_{reg} = \sum_{x \in \mathbf{Q} - \{0,1\}} I(x),$$

where, for a certain test function f whose local components will be recalled below,

$$I(x) = \iint_{\mathbf{A}^* \times \mathbf{A}^*} f(\begin{pmatrix} ab & ax \\ b & 1 \end{pmatrix}) \chi(a)^{-1} d^*a\, d^*b.$$

Here, we abuse notation and write χ for the unitary adelic Hecke character determined by the Dirichlet character χ fixed earlier. The integrals $I(x)$ are computed locally in [RR1, §2.7] and their sum is bounded in §3 of their paper. We need to reexamine these proofs in order to determine the dependence on p. At the same time, we will also keep track of the dependence on the weight k. The final result is given in Theorem 7.5.

The statements of [RR1, Prop. 2.4abcde] each have errors, but this does not affect the validity of the trace formula given in §5 of their paper. The following is a corrected version of that proposition.

Proposition 7.2. *For $x \in \mathbf{Q} - \{0,1\}$ and f_v as in* [RR1]*, define the local integrals*

$$I_v(x) = \iint_{\mathbf{Q}_v^* \times \mathbf{Q}_v^*} f_v(\begin{pmatrix} ab & ax \\ b & 1 \end{pmatrix}) \chi_v(a)^{-1} d^*a\, d^*b.$$

Then the following statements hold.

(a) *Let $v = q$ be a finite prime not dividing pND. Then:*
- $I_v(x) = 0$ *if* $v(1-x) > 0$.
- *If $v(1-x) = 0$ and $v(x) = 0$, then $I_v(x) = 1$.*
- *Generally if $v(1-x) \le 0$, then*

$$|I_v(x)| \le \begin{cases} v(x)^2 & \text{if } v(x) \neq 0 \\ 1 & \text{if } v(x) = 0. \end{cases}$$

(b) *Let $v = q$ be a prime dividing D, and write $c = v(D) \ge 1$. Then:*
- $I_v(x) = 0$ *if* $v(1-x) > c$.
- *If $v(1-x) \le c$, then*

$$|I_v(x)| \le 6q^{c/2}(2c + 1 + |v(x)|) \le 6q^{c/2}(2c+1)(1 + |v(x)|).$$

(c) *Let $v = N$. Then $I_v(x) = 0$ unless $v(x) \ge 1$ (and hence $v(1-x) = 0$). In this case*

$$|I_N(x)| \le \nu(N) |v_N(x)|.$$

(d) *Let $v = p$, and let f_p be the test function defined in* (7.2)*. We suppose $\ell > 0$ since the $\ell = 0$ case is covered by (a). Then $I_p(x)$ vanishes unless $v(1-x) \le \ell$, in which case*

$$|I_p(x)| \le 4p^{-\ell/2} \ell(\ell + 1 + |v(x)|) \le 4p^{-\ell/2} \ell(\ell+1)(1 + |v(x)|).$$

(e) *When $v = \infty$,*

$$|I_\infty(x)| \ll \frac{|1-x|^{k/2}}{|x|}$$

for an absolute implied constant.

Proof. We follow the proof and notation of [RR1]. We begin with part (e), where $f_\infty(g) = d_k\overline{\langle \pi_k(g)v, v\rangle}$ is the matrix coefficient of the weight k discrete series representation of $\mathrm{PGL}_2(\mathbf{R})$ with lowest weight unit vector v and formal degree d_k. In [RR1, Prop 2.4e], $I_\infty(x)$ is expressed in terms of a certain quantity $I_\infty(\varepsilon, \delta, \nu)$ which is defined as being independent of x. This seems to be a typo; as is clear from their proof, $I_\infty(x)$ does depend on x. But the proof is flawed anyhow for other reasons, so we will not try to correct the definition of $I_\infty(\varepsilon, \delta, \nu)$. For $\delta, \nu \in \{\pm 1\}$, set

$$I'_x(\delta, \nu) = \int_0^\infty \int_0^\infty \frac{a^{k/2-1} b^{k/2-1} da\, db}{(ax - \nu b + \delta i(ab + \nu))^k}.$$

(This is $I'_\infty(-\nu, \delta, \nu)$ in the notation of [RR1].) Following the proof in [RR1] (we caution that the displayed formula there for $f_\infty(\left(\begin{smallmatrix} ab & ax \\ b & 1 \end{smallmatrix}\right))$ is incorrect), we find, upon observing that $(-1)^k = 1$ since k must be even, that

$$I_\infty(x) = \begin{cases} d_k(2i)^k(1-x)^{k/2}[I'_x(-1,1) - I'_x(1,1)] & \text{if } 1 - x > 0 \\ d_k(2i)^k(x-1)^{k/2}[I'_x(-1,-1) - I'_x(1,-1)] & \text{if } 1 - x < 0. \end{cases} \tag{7.4}$$

As shown in the proof of [RR1, Lemma 7], we have

$$I'_x(\delta, \nu) = B(\tfrac{k}{2}, \tfrac{k}{2})(\delta i)^{k/2} J_x, \tag{7.5}$$

where $B(x, y)$ is the Beta function, and

$$J_x = \int_0^\infty \frac{a^{k/2-1} da}{(ax + \delta\nu i)^{k/2}(a + \delta\nu i)^{k/2}}.$$

The proof in [RR1] now rotates the line of integration to a purely imaginary ray, overlooking the fact that this ray passes through poles of the integrand in many cases. (Their proof is fixable if one assumes $x > 0$, but in fact $I_\infty(x)$ need not vanish if $x < 0$, despite the assertion to the contrary in [RR1, §3].) The integral J_x can presumably be computed in terms of special functions even when $x < 0$, but since ultimately this integral forms part of an error term, we choose simply to bound it as follows. Observing that $|\frac{a}{a \pm i}| < 1$ for $a > 0$,

$$\begin{aligned} |J_x| \le \int_0^\infty \frac{da}{|ax \pm i|^{k/2}|a \pm i|} &\le \int_0^\infty \frac{da}{|ax \pm i|^{k/2}} = \frac{1}{|x|}\int_0^\infty \frac{du}{|u \pm i|^{k/2}} \\ &= \tfrac{1}{2} B(\tfrac{1}{2}, \tfrac{k}{4} - \tfrac{1}{2})|x|^{-1} \end{aligned}$$

by [GR, 8.380.3]. By the above, (7.4), (7.5), and noting that for the standard measure used in [RR1], $d_k = \frac{k-1}{4\pi}$ (cf. [KL1, Prop. 14.4]), we have

$$|I_\infty(x)| \ll 2^k k B(\tfrac{1}{2}, \tfrac{k-2}{4}) B(\tfrac{k}{2}, \tfrac{k}{2}) \frac{|1-x|^{k/2}}{|x|}$$

for an absolute implied constant. By Stirling's formula, $B(\frac{k}{2}, \frac{k}{2}) \sim \frac{2\sqrt{\pi}}{\sqrt{\frac{k}{2}}2^k}$, and $B(\frac{1}{2}, \frac{k-2}{4}) \sim \frac{2\sqrt{\pi}}{(k-2)^{1/2}}$. This gives $|I_\infty(x)| \ll |1-x|^{k/2}|x|^{-1}$ for an absolute implied constant, which proves assertion (e).

To prove (a) and (d), let q be a prime, fix an integer $r \ge 0$, and let f_q be the characteristic function of $Z_q K_q \left(\begin{smallmatrix} q^r & \\ & 1 \end{smallmatrix}\right) K_q$. Then $f_q(\left(\begin{smallmatrix} ab & ax \\ b & 1 \end{smallmatrix}\right))$ is nonzero if and only if there exists $\lambda \in \mathbf{Q}_q^*$ such that $\left(\begin{smallmatrix} \lambda ab & \lambda ax \\ \lambda b & \lambda \end{smallmatrix}\right) \in K_q \left(\begin{smallmatrix} q^r & \\ & 1 \end{smallmatrix}\right) K_q$. By the theory of determinantal divisors ([Ne, p. 28]), a matrix $g \in G(\mathbf{Q}_q)$ belongs to $K_q \begin{pmatrix} q^r & \\ & 1 \end{pmatrix} K_q$ if and only if each of the following holds:

- $\det g \in q^r \mathbf{Z}_q^*$
- each entry of g belongs to $\mathbf{Z}_q$
- some entry of g belongs to $\mathbf{Z}_q^*$.

(When $r = 0$, the third condition is already implied by the first.) Therefore, $f_q(\left(\begin{smallmatrix} ab & ax \\ b & 1 \end{smallmatrix}\right)) \neq 0$ if and only if there exists $\lambda \in \mathbf{Q}_q^*$ such that:

(1) $2v(\lambda) + v(a) + v(b) + v(1-x) = r$
(2) $v(\lambda) + v(a) + v(b) \geq 0$
(3) $v(\lambda) + v(a) + v(x) \geq 0$
(4) $v(\lambda) + v(b) \geq 0$
(5) $v(\lambda) \geq 0$
(5b) Equality occurs in at least one of (2)-(5).

Eliminating $v(\lambda)$, we obtain the following conditions:

(6) $v(a) + v(x) - v(1-x) \geq -r$ (from (2)+(3)−(1))
(7) $v(b) - v(1-x) \geq -r$ (from (2)+(4)−(1))
(8) $v(x) - v(1-x) \geq -r$ (from (3)+(4)−(1))
(9) $v(1-x) \leq r$ (from (1)−(2)−(5))
(10) $v(a) + v(1-x) \leq r$ (from (1)−(4)−(5))
(11) $v(b) + v(1-x) - v(x) \leq r$ (from (1)−(3)−(5))
(12) $v(a) + v(b) + v(1-x) \leq r$ (from (1)−2(5))
(13) $v(b) \geq v(a) + v(1-x) - r$ (from 2(4)−(1)).

This leads to the following condensed set of conditions, the last of which is from (5b) and was overlooked in the proof of [RR1, Prop. 2.4]:

(i) $v(1-x) \leq r$
(ii) $v(x) \geq v(1-x) - r$
(iii) $v(1-x) - v(x) - r \leq v(a) \leq \min\{r - v(1-x), r - v(1-x) - v(b)\}$
(iv) $\max\{v(1-x) - r, v(a) + v(1-x) - r\} \leq v(b) \leq v(x) + r - v(1-x)$
(v) At least one of the following holds:
  (va) $v(a) + v(b) + v(1-x) = r$ (if $v(\lambda) = 0$, using (1))
  (vb) $v(a) + v(b) - v(1-x) = -r$ (if (2)=0, using 2(2)−(1))
  (vc) $v(a) - v(b) + 2v(x) - v(1-x) = -r$ (if (3)=0, using 2(3)−(1))
  (vd) $v(b) - v(a) - v(1-x) = -r$ (if (4)=0, using 2(4)−(1))

We may now prove part (a). Suppose $q \nmid pND$. Then f_q is the characteristic function of K_q and we can take $r = 0$ in the above discussion. The first part of (a) follows from (i). If $r = v(x) = v(1-x) = 0$, we see from (iii) and (iv) that $v(a) = v(b) = 0$, and since χ_q is unramified and $\text{meas}(\mathbf{Z}_q^*) = 1$, it follows that $I_v(x) = 1$. Now suppose $v(1-x) < 0$. Then $v(x) = v(1-x)$, and (iii) and (iv) become

$$0 \leq v(a) \leq -v(x), \qquad v(x) \leq v(b) \leq 0.$$

Using the fact that χ_q is unramified and $\text{meas}(\mathbf{Z}_q^*) = 1$, we find

$$|I_v(x)| \leq \sum_{m=0}^{-v(x)} \sum_{n=v(x)}^{0} 1,$$

and the last assertion of (a) follows in this case. Likewise, if $v(1-x)=0$, then $v(x)\geq 0$, and (iii) and (iv) become

$$-v(x)\leq v(a)\leq 0, \qquad 0\leq v(b)\leq v(x),$$

and the assertion holds in this case as well. This proves (a).

Before proving (d), we make some observations about the above conditions for general $r\geq 0$. If $v(1-x)\leq r$, we see from (v) that once $v(a)$ is fixed, there are at most *four* possibilities for $v(b)$. Setting $m=v(a)$ and $n=v(b)$, we immediately see that

$$|I_v(x)|\leq \sum_{m=v(1-x)-r-v(x)}^{r-v(1-x)} \sum_{n\in\{4\text{ values}\}} 1 = 4\Big(2r-2v(1-x)+v(x)+1\Big).$$

Observing that if $v(1-x)>0$ (resp. $v(1-x)=0$, resp. $v(1-x)<0$) then $v(x)=0$ (resp. $v(x)\geq 0$, resp. $v(x)=v(1-x)$), it follows easily that in all cases,

$$|I_v(x)|\leq 4\Big(2r+1+|v(x)|\Big)\leq 8(r+1+|v(x)|). \tag{7.6}$$

Now suppose $q=p$ and f_p is the test function defined in (7.2). Then by the above, $I_p(x)$ vanishes if $v(1-x)>\ell$. When $v(1-x)\leq \ell$, by (7.1), (7.2) and (7.6), we have

$$|I_p(x)|\leq p^{-\ell/2}\sum_{j=0}^{\lfloor \frac{\ell}{2}\rfloor} 8(\ell-2j+1+|v(x)|)\leq p^{-\ell/2}\tfrac{8\ell}{2}(\ell+1+|v(x)|).$$

This proves (d).

Next, consider $v=N$. Then for

$$K_0(N)_N=\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathbf{Z}_N)|\, c\in N\mathbf{Z}_N\},$$

f_N is the characteristic function of $Z_N K_0(N)_N$, scaled by $\nu(N)$. So $f_N(\left(\begin{smallmatrix} ab & ax\\ b & 1\end{smallmatrix}\right))\neq 0$ if and only if there exists $\lambda\in\mathbf{Q}_N^*$ such that $\left(\begin{smallmatrix} \lambda ab & \lambda ax\\ \lambda b & \lambda\end{smallmatrix}\right)\in K_0(N)_N$. The lower right entry must be a unit, which means that in fact we may take $\lambda=1$. Therefore

$$\left(\begin{smallmatrix} ab & ax\\ b & 1\end{smallmatrix}\right)\in K_0(N)_N,$$

which means:

(1″) $v(a)+v(b)+v(1-x)=0$
(2″) $v(a)+v(b)=0$
(3″) $v(a)+v(x)\geq 0$
(4″) $v(b)\geq 1$.

As a result, the integrand vanishes unless:

- $v(1-x)=0$
- $v(a)=-v(b)\leq -1$
- $v(x)\geq 1$.

It follows that $I_N(x)=0$ unless $v(x)\geq 1$, in which case

$$|I_N(x)|\leq \sum_{m=-v(x)}^{-1}\nu(N),$$

which proves (c).

Lastly, take $v=q$ to be a prime divisor of D, and set $c=v(D)\geq 1$. There are some oversights in the definition of the local test function f_q at such a place in [RR1, p. 706]: the notation $\chi_{1,v}$ is not defined, χ_v does not define a character of the additive group X, and it is asserted that the integral $g(\chi_v)$ defined there, which clearly has absolute value ≤ 1, coincides with the classical Gauss sum which has absolute value $q^{c/2}$. A detailed treatment of the local test function with the desired spectral properties (and giving the same main term on the geometric side in [RR1]) is given in [JK, (3.11)-(3.12)]. For our purpose, it is enough to know that

$$\mathrm{Supp}(f_q) = \bigcup_{\substack{m \bmod D\mathbf{Z}_q \\ q \nmid m}} \left(\begin{smallmatrix} 1 & -m/D \\ 0 & 1 \end{smallmatrix}\right) Z_q K_q,$$

and $f_q = \sum_m f_{m,q}$, where $f_{m,q}$ is supported on the coset indexed by m and has absolute value $q^{-c/2}$ there.

To match the notation in [RR1], let $z = m/D$ (so $v(z) = -c$) and write $f_{z,v}$ for $f_{m,q}$. Then $f_{z,v}(\left(\begin{smallmatrix} ab & ax \\ b & 1 \end{smallmatrix}\right)) \neq 0$ if and only if there exists $\lambda \in \mathbf{Q}_q^*$ such that

$$\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} ab & ax \\ b & 1 \end{pmatrix} = \begin{pmatrix} \lambda b(a+z) & \lambda(ax+z) \\ \lambda b & \lambda \end{pmatrix} \in K_q.$$

Thus,

(1′) $2v(\lambda) + v(a) + v(b) + v(1-x) = 0$
(2′) $v(\lambda) + v(a+z) + v(b) \ge 0$
(3′) $v(\lambda) + v(ax+z) \ge 0$
(4′) $v(\lambda) + v(b) \ge 0$
(5′) $v(\lambda) \ge 0$
(5′b) Equality holds in at least one of (2′)-(5′).

As before, we eliminate $v(\lambda)$ to get the following:

(6′) $v(a+z) + v(ax+z) - v(a) - v(1-x) \ge 0$ (from (2′)+(3′)−(1′))
(7′) $v(b) + v(a+z) - v(a) \ge v(1-x)$ (from (2′)+(4′)−(1′))
(8′) $v(ax+z) - v(a) - v(1-x) \ge 0$ (from (3′)+(4′)−(1′))
(9′) $v(1-x) \le v(a+z) - v(a)$ (from (1′)−(2′)−(5′))
(10′) $v(a) + v(1-x) \le 0$ (from (1′)−(4′)−(5′))
(11′) $v(a) + v(b) + v(1-x) - v(ax+z) \le 0$ (from (1′)−(3′)−(5′))
(12′) $v(a) + v(b) + v(1-x) \le 0$ (from (1′)−2(5′))
(13′) $v(b) \ge v(a) + v(1-x)$ (from 2(4′)−(1′)).

(Only (11′) differs from the list in [RR1], whose (11′) seems to be an unmodified paste from (11).) We claim that the above implies the following set of conditions:

(x) $v(1-x) \le c$
(y) $v(1-x) - c \le v(b) \le v(x) + c - v(1-x)$
(z) At least one of the following holds:
  (zi) $v(a) = -v(1-x) - v(b)$ (if $v(\lambda) = 0$, using (1′))
  (zii) $v(a) + v(1-x) - 2v(a+z) - v(b) = 0$ (if (2′)=0, using (1′)−2(2′))
  (ziii) $v(a) + v(b) + v(1-x) - 2v(ax+z) = 0$ (if (3′)=0, using (1′)−2(3′))
  (ziv) $v(a) - v(b) + v(1-x) = 0$ (if (4′)=0, using (1′)−2(4′)).

It suffices to prove (x) and (y) since (z) follows from $(5'b)$. To prove (x), if $v(a) \neq v(z)$, then $v(a+z) = \min\{v(a), v(z)\}$, so $v(a+z) - v(a) \le 0$, which, by (9′), gives $v(1-x) \le 0 < c$. On the other hand, if $v(a) = v(z) = -c$, then by (10′), $v(1-x) \le c$, as needed.

For (y), note that if $v(a) = v(z) = -c$, then (13′) gives $v(1-x) - c \le v(b)$ in that case. If $v(a) \neq v(z)$, then as before $v(a+z) - v(a) \le 0$, and (7′) then gives $v(1-x) - c < v(1-x) \le v(b)$. This proves the lower bound in (y). For the upper bound, suppose first that $v(ax) \neq v(z)$. Then $v(ax+z) = \min\{v(ax), v(z)\}$, so $v(ax+z) \leq v(a) + v(x)$. (11′) then gives $v(b) \leq v(x) - v(1-x)$, which is stronger than the desired upper bound. If $v(ax) = v(z)$, then (11′) is not helpful because $v(ax+z) = \infty$ is possible. However, in this case $v(ax) = v(a) + v(x) = -c$, so (12′) gives $v(b) \leq v(x) + c - v(1-x)$, as needed.

Finally, we claim that once $v(b)$ is fixed, there are at most *six* possible values of $v(a)$ for which (z) is satisfied. It suffices to show that there are at most two possibilities for $v(a)$ if (zii) (resp. (ziii)) is satisfied. Suppose a and $\tilde{a}$ have different valuations and each satisfy (zii). We claim that $v(\tilde{a}) = -v(a)$. Write $\tilde{a} = q^t ua$ for $u \in \mathbf{Z}_q^*$ and some integer $t \neq 0$. Then

$$v(\tilde{a}) - 2v(\tilde{a}+z) = v(a) - 2v(a+z),$$

which gives

$$v(q^t ua + z) = v(a+z) + \tfrac{t}{2}.$$

By Lemma 7.3 below, we get $t = -2v(a)$, as claimed. For (ziii), by the same argument we get

$$v(q^t uax + z) = v(ax+z) + \tfrac{t}{2},$$

so t is again determined by Lemma 7.3: $t = -2v(ax)$.

By the above discussion, summing over z (i.e. over $m \in (\mathbf{Z}_q/D\mathbf{Z}_q)^*$), and using $|f_{z,v}(g)| = q^{-c/2}$ if nonzero, when $v(1-x) \leq c$ we have

$$\begin{aligned} |I_q(x)| \leq q^{-c/2}\varphi(q^c) \sum_{n=v(1-x)-c}^{v(x)+c-v(1-x)} \sum_{\{6 \text{ values}\}} 1 &\leq q^{c/2} 6\big(v(x) + 2c - 2v(1-x) + 1\big) \\ &\leq 6q^{c/2}(2c + 1 + |v(x)|), \end{aligned}$$

where the latter inequality is obtained by considering the cases $v(1-x)$ being greater than, equal to, or less than 0. This proves part (b) of the proposition. □

Lemma 7.3. *Let $a, z \in \mathbf{Q}_q^*$ with $a + z \neq 0$, and suppose there exist $u \in \mathbf{Z}_q^*$ and t a nonzero integer such that*

$$v(q^t ua + z) = v(a+z) + \tfrac{t}{2} \tag{7.7}$$

where $v = v_q$. Then $t = -2v(a)$.

Proof. By pulling powers of q aside, we may reduce to the case where $z \in \mathbf{Z}_q^*$. If $v(a) = 0$ too, then $v(a+z) \geq 0$, and (7.7) leads to a contradiction if either $t > 0$ or $t < 0$. Suppose $v(a) > 0$, so that $v(a+z) = 0$. If $t > -v(a)$, then (7.7) becomes

$$0 = v(q^t ua + z) = \tfrac{t}{2},$$

a contradiction. If $t = -v(a)$, then (7.7) becomes

$$0 \leq v(q^t ua + z) = \tfrac{t}{2}$$

which is also a contradiction. If $t < -v(a)$, then (7.7) becomes

$$t + v(a) = v(q^t ua + z) = \tfrac{t}{2},$$

which gives $t = -2v(a)$. A similar analysis gives the same conclusion if $v(a) < 0$. □

Proposition 7.4. *With local components f_v as in Proposition 7.2, the sum of the regular terms is*

$$I_{reg} \ll \frac{\nu(N)D^{k/2}}{N^{k/2-\varepsilon}} p^{\ell(\frac{k+1}{2}+\varepsilon)},$$

for any $\varepsilon > 0$, where the implied constant depends only on ℓ and D.

Proof. We closely follow [RR1, §3]. Let $M = Dp^\ell$. Suppose $I(x) \neq 0$. Then by Proposition 7.2, $v_q(1-x) \le v_q(M)$ for all primes q. This means that $n := \frac{M}{1-x} \in \mathbf{Z}$. The map $x \mapsto \frac{1}{1-x}$ is a bijection from $\mathbf{Q} - \{0,1\}$ to itself. Therefore n is not equal to 0 or M. Since $N \nmid M$ and $v_N(1-x) = 0$ by Proposition 7.2c, we have

$$v_N(n-M) = v_N(M(\tfrac{1}{1-x} - 1)) = v_N(\tfrac{x}{1-x}) = v_N(x) \ge 1,$$

where the latter inequality is again from Proposition 7.2c. Thus $N|(n-M)$. Note that $x = \frac{n-M}{n}$. So

$$I_{reg} = \sum_{\substack{n \in M+N\mathbf{Z},\\ n\neq 0,M}} I\Big(\frac{n-M}{n}\Big). \tag{7.8}$$

Since $N \nmid M$, the condition $n \neq 0$ is superfluous. As mentioned earlier, the assertion in [RR1, §3] that $I_\infty(x) = 0$ if $x < 0$ is incorrect. Now by Proposition 7.2,

$$I(\tfrac{n-M}{n}) \ll p^{-\ell/2}\sqrt{D}\nu(N)|I_\infty(\tfrac{n-M}{n})| \prod_{q|n(n-M)} B_q(\tfrac{n-M}{n}), \tag{7.9}$$

where

$$B_q(\tfrac{n-M}{n}) = \begin{cases} 1 & \text{if } v_q(\frac{n-M}{n}) = 0 \text{ and } q \nmid pDN \\ v_q(\frac{n-M}{n})^2 & \text{if } v_q(\frac{n-M}{n}) \neq 0 \text{ and } q \nmid pDN \\ |v_q(\frac{n-M}{n})| & \text{if } q = N \\ (1 + |v_q(\frac{n-M}{n})|) & \text{if } q|M(= p^\ell D), \end{cases}$$

and the implied constant in (7.9) depends only on ℓ and D.

For the archimedean part, by Proposition 7.2e we have

$$|I_\infty(\tfrac{n-M}{n})| \ll |1 - \tfrac{n-M}{n}|^{k/2} \cdot |\tfrac{n-M}{n}|^{-1} = \frac{M^{k/2}}{|n|^{k/2}|1 - \frac{M}{n}|}.$$

Observe that for fixed M, $|1 - \frac{M}{n}|$ is as small as possible when $n = M+1$ since $n \neq M$. Hence $|1 - \frac{M}{n}| \ge \frac{1}{M+1}$. So for an absolute implied constant,

$$|I_\infty(\tfrac{n-M}{n})| \ll \frac{M^{k/2+1}}{|n|^{k/2}}. \tag{7.10}$$

For the product in (7.9), writing $\frac{n-M}{n} = \prod_{q|n(n-M)} q^{b_q}$, for any $\varepsilon > 0$ we have

$$|v_q(\tfrac{n-M}{n})| = |b_q| = \log_q(q^{|b_q|}) \le Cq^{\varepsilon|b_q|}$$

for a positive constant C independent of q. Therefore letting $\omega(n) \sim \log\log n$ denote the number of distinct prime factors of n, for any sufficiently small $\varepsilon > 0$ which may not be the same each time we use it, we have

$$\prod_{q|n(n-M)} |v_q(\tfrac{n-M}{n})| \ll C^{\omega(n(n-M))} \prod_{q|n(n-M)} q^{\varepsilon|b_q|} \ll |n|^\varepsilon|n-M|^\varepsilon$$
$$\ll |n|^\varepsilon|nM|^\varepsilon \ll |n|^\varepsilon M^\varepsilon.$$

It follows similarly that

$$\prod_{q|n(n-M)} B_q(\tfrac{n-M}{n}) \ll |n|^{\varepsilon} M^{\varepsilon} \tag{7.11}$$

for any $\varepsilon > 0$.

Using (7.10) and (7.11) and recalling that $M = p^{\ell} D$, (7.9) gives

$$|I(\tfrac{n-M}{n})| \ll_{D,\ell} p^{-\frac{\ell}{2}} p^{\ell(\frac{k}{2}+1+\varepsilon)} D^{k/2} \nu(N) \frac{1}{|n|^{k/2-\varepsilon}}.$$

So

$$I_{reg} \ll p^{\ell(\frac{k+1}{2}+\varepsilon)} D^{k/2} \nu(N) \sum_{\text{nonzero } m\in\mathbf{Z}} \frac{1}{|M+Nm|^{k/2-\varepsilon}}.$$

We can pull $\frac{1}{N^{k/2-\varepsilon}}$ out of the sum, and what remains is bounded by an absolute constant since $\frac{M}{N} \notin \mathbf{Z}$ and $k > 2$. This gives

$$I_{reg} \ll \nu(N) D^{k/2} \frac{p^{\ell(\frac{k+1}{2}+\varepsilon)}}{N^{k/2-\varepsilon}}. \qquad \square$$

By what we have shown, along with the computation of the main term and measure in [RR1], upon dividing through by $\nu(N)$ we obtain the following.

Theorem 7.5. *Let $k > 2$ be an even integer, $\chi = \chi_{-D}$ be as in Theorem 1.2, N a prime not dividing D with $\chi(-N) = -1$, and p a prime not dividing ND. Then for all $\ell \geq 0$ and $\varepsilon > 0$,*

$$\frac{1}{\nu(N)} \sum_{f\in\mathcal{F}_{N,k}^{new}} \frac{\Lambda(\frac{1}{2}, f)\Lambda(\frac{1}{2}, f\times\chi)}{\|f\|^2} X_{\ell}(\lambda_f(p))$$
$$= 2c_k L(1,\chi) \int_{-\infty}^{\infty} X_{\ell}\, d\eta_p + O\Big(\frac{p^{\ell(\frac{k+1}{2}+\varepsilon)} D^{k/2}}{N^{k/2-\varepsilon}}\Big)$$

where $c_k = d_k 2^k B(\frac{k}{2}, \frac{k}{2})$, and the implied constant depends only on ℓ and D.

Remarks: (1) This extends the trace formula given in [RR1, §5] by providing the dependence on p and k in the error term. We note that our Petersson norm (2.1) has a factor of $\nu(N)^{-1}$ not present in that of [RR1].

(2) If $\int X_{\ell}\, d\eta_p \neq 0$, it follows that the sum on the left-hand side is nonzero whenever $N > D$ and $N + k$ is sufficiently large relative to ℓ. This can be seen using (7.12) below.

Proof of Proposition 5.1. By Theorem 7.5, we have

$$\frac{1}{\nu(N)} \sum_{f\in\mathcal{F}} w_f X_{\ell}(\lambda_f(p)) = F_{\ell} + E_{\ell},$$

where F_{ℓ} is the main term and $E_{\ell} \ll p^{\ell(\frac{k+1}{2}+\varepsilon)} C_0$, where $C_0 = \frac{D^{k/2}}{N^{k/2-\varepsilon}}$. By the proof of Proposition 3.1,

$$\frac{\sum_{f\in\mathcal{F}} w_f X_{\ell}(\lambda_f(p))}{\sum_{f\in\mathcal{F}} w_f} = \int_{-\infty}^{\infty} X_{\ell}\, d\eta_p + O\Big(p^{\ell(\frac{k+1}{2}+\varepsilon)} \frac{\frac{C_0}{F_0}}{1+\frac{E_0}{F_0}}\Big).$$

(Note that $F_0 \neq 0$ since $\int X_0 \, d\eta_p = 1$ as shown in Proposition 6.1.) As noted earlier, $2^k B(\frac{k}{2}, \frac{k}{2}) \sim \frac{2\sqrt{2\pi}}{\sqrt{k}}$, so that

$$c_k = \frac{k-1}{4\pi} 2^k B(\tfrac{k}{2}, \tfrac{k}{2}) \sim \sqrt{\frac{k}{2\pi}}. \tag{7.12}$$

Now

$$\frac{E_0}{F_0} \ll \frac{C_0}{F_0} = \frac{D^{k/2}}{2c_k L(1,\chi) N^{k/2-\varepsilon}} \ll \frac{D^{k/2}}{k^{1/2} N^{k/2-\varepsilon}},$$

and Proposition 5.1 follows. □

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