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Matrix coefficients of depth-zero supercuspidal representations of GL(2)

Andrew Knightly and Carl Ragsdale

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We give explicit formulas for matrix coefficients of the depth-zero supercuspidal representations of GL(2) over a nonarchimedean local field, highlighting the case where the test vector is a unit new vector. We also describe the partition of the set of such representations according to central character, and compute sums of matrix coefficients over all representations in a given class.

Introduction

Let F be a nonarchimedean local field with integer ring $\mathfrak{o}$, maximal ideal $\mathfrak{p} = \varpi\mathfrak{o}$, and residue field $k = \mathfrak{o}/\mathfrak{p}$ of cardinality q. The supercuspidal representations of $\mathrm{GL}_2(F)$ are precisely those irreducible admissible representations which do not arise as constituents of parabolic induction. They are characterized by having matrix coefficients which are compactly supported modulo the center.

In this paper we explicitly compute the matrix coefficients of depth-zero supercuspidal representations. These are the supercuspidals with the smallest possible conductor exponent, namely 2. First discovered by Mautner [1964, Section 9], they arise by compact induction from the $(q-1)$-dimensional representations of $\mathrm{GL}_2(\mathfrak{o})$ inflated from the cuspidal series of the finite group $\mathrm{GL}_2(k)$.

In the first section, we show that the matrix coefficients of any supercuspidal representation are expressible in terms of those of the finite-dimensional inducing representation. Thus, the task at hand essentially reduces to a computation of the matrix coefficients of the cuspidal representations of $\mathrm{GL}_2(k)$. The latter is achieved in Theorem 2.7 using the explicit model from [Piatetski-Shapiro 1983].

With global applications in mind, in Section 3 we single out the case where the test vector in the supercuspidal matrix coefficient is a unit new vector. The resulting function, given in (3-6) and Theorem 3.2, may be used to define an integral operator on the global automorphic spectrum of GL_2 which isolates those cuspidal

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newforms with depth-zero supercuspidal local type (see Section 3.4). Possible applications include various trace formulas involving newforms of level p^2 which are supercuspidal (as opposed to principal series or special) at p. For a recent example involving the supercuspidal representations of conductor p^3, see [Knightly and Li 2012].

It is often desirable to organize representations according to central character. For example, there are exactly $2(q-1)$ supercuspidal representations of $\mathrm{GL}_2(F)$ of conductor p^3 with a given central character (see [Bushnell and Henniart 2014, Remark 2.2]). By contrast, there are $(q/2)(q-1)$ distinct cuspidal representations of $\mathrm{GL}_2(k)$ and $(q-1)$ possible central characters, but obviously there cannot be $q/2$ of each kind if q is odd. We sort this out in Proposition 2.3, and use it to give a formula (4-1) for the number of supercuspidals of conductor p^2 with a given central character. It depends on the parity of q and the order of the central character. We then give formulas for various sums of matrix coefficients over the set of depth-zero supercuspidal representations with a given central character. These computations rely on sum formulas for primitive characters, derived in Proposition 2.4. We close in the final section with some simple examples.

1. Matrix coefficients of supercuspidal representations

In this section let $G = \mathrm{GL}_n(F)$, or more generally, any unimodular locally profinite group [Bushnell and Henniart 2006] with center Z. Let $H \subset G$ be an open and closed subgroup containing Z with H/Z compact, and let (ρ, V) be an irreducible smooth representation of H. Consider the compact induction $\pi = \text{c-Ind}_H^G(\rho)$. It consists of the functions $\phi : G \longrightarrow V$ with compact support (mod Z) for which $\phi(hg) = \rho(h)\phi(g)$ for all $h \in H$, $g \in G$, with G acting on the space by right translation. Here we show that, as observed by Mautner [1964], the matrix coefficients of π are essentially those of ρ. These matrix coefficients are compactly supported (modulo Z), so if π is irreducible and admissible, it is supercuspidal. Conversely, it is conjectured that all supercuspidal representations arise in this way. This was proven by Bushnell and Kutzko [1993] for $G = \mathrm{GL}_n(F)$, and more recently in great (but not complete) generality in [Stevens 2008; Kim 2007].

We assume for simplicity that ρ has unitary central character, so that by the fact that H/Z is compact, ρ is unitarizable. Let $\langle v, w\rangle_V$ denote an H-equivariant inner product on V. Then the inner product on $\text{c-Ind}_H^G(\rho)$ given by

$$\langle \phi, \psi\rangle = \sum_{x\in H\backslash G} \big\langle \phi(x), \psi(x)\big\rangle_V$$

is convergent (in fact a finite sum) and well-defined. Further, for any $g \in G$,

$$\big\langle \pi(g)\phi, \pi(g)\psi\big\rangle = \sum_{x\in H\backslash G} \big\langle \phi(xg), \psi(xg)\big\rangle_V = \sum_{x\in H\backslash G} \big\langle \phi(x), \psi(x)\big\rangle_V = \langle \phi, \psi\rangle.$$

Thus π is unitary relative to this inner product.

For $v \in V$ and $y \in G$, define a function $f_{y,v} \in \text{c-Ind}_H^G(\rho)$ by

$$f_{y,v}(g) = \begin{cases} \rho(h)v & \text{if } g = hy \in Hy, \\ 0 & \text{if } g \notin Hy. \end{cases}$$

Then the set $\{f_{y,v} \mid y \in H\backslash G, v \in V\}$ spans the space $\text{c-Ind}_H^G(\rho)$. (Note that $f_{hy,v} = f_{y,\rho(h^{-1})v}$.)

Proposition 1.1. *For $y, z \in G$ and $v, w \in V$,*

$$\langle \pi(g) f_{y,v}, f_{z,w} \rangle = \begin{cases} \langle \rho(h)v, w \rangle_V & \text{if } g = z^{-1}hy \in z^{-1}Hy, \\ 0 & \text{if } g \notin z^{-1}Hy. \end{cases}$$

Proof. By definition of the inner product,

$$\begin{aligned} \langle \pi(g) f_{y,v}, f_{z,w} \rangle &= \sum_{x \in H\backslash G} \langle \pi(g) f_{y,v}(x), f_{z,w}(x) \rangle_V \\ &= \sum_{x \in H\backslash G} \langle f_{y,v}(xg), f_{z,w}(x) \rangle_V \\ &= \langle f_{y,v}(zg), w \rangle_V, \end{aligned}$$

since $f_{z,w}(x)$ vanishes unless $x \in Hz$. If $g = z^{-1}hy \in z^{-1}Hy$, then the above is equal to

$$\langle f_{y,v}(hy), w \rangle_V = \langle \rho(h)v, w \rangle_V,$$

as needed. If $g \notin z^{-1}Hy$, then $zg \notin Hy$, so $f_{y,v}(zg) = 0$ and the inner product vanishes. □

If we let $\overline{G} = G/Z$, then the formal degree d_π of π is a positive constant satisfying

$$\int_{\overline{G}} |\langle \pi(g) f, f \rangle|^2 dg = \frac{\|f\|^4}{d_\pi} \tag{1-1}$$

for all $f \in \text{c-Ind}_H^G(\rho)$. It depends on the choice of Haar measure on $\overline{G}$. (The existence of d_π is due to Godement; see, for example, [Knightly and Li 2006, Proposition 10.4].)

Proposition 1.2. *For any choice of Haar measure on $\overline{G} = G/Z$, the associated formal degree of π is given by*

$$d_\pi = \frac{\dim \rho}{\text{meas}(\overline{H})},$$

where $\overline{H}$ is the (open) image of H in $\overline{G}$.

Proof. Let $v \in V$ be a unit vector, and consider the function $f_{1,v}$. By Proposition 1.1,

$$\langle \pi(g) f_{1,v}, f_{1,v} \rangle = \begin{cases} \langle \rho(g)v, v \rangle_V & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

Therefore,

$$\begin{aligned} \int_{\overline{G}} \left| \langle \pi(g) f_{1,v}, f_{1,v} \rangle \right|^2 dg &= \int_{\overline{H}} \left| \langle \rho(g)v, v \rangle_V \right|^2 dg \\ &= \frac{\|v\|^4}{\dim(\rho)} \operatorname{meas}(\overline{H}) = \frac{\operatorname{meas}(\overline{H})}{\dim(\rho)}, \end{aligned}$$

by the Schur orthogonality relations for irreducible representations of compact groups. By (1-1),

$$\frac{\operatorname{meas}(\overline{H})}{\dim(\rho)} = \frac{\|f_{1,v}\|^4}{d_\pi}.$$

Therefore, it suffices to show that $\|f_{1,v}\| = 1$. This can be done via a direct computation:

$$\begin{aligned} \|f_{1,v}\|^2 = \langle f_{1,v}, f_{1,v} \rangle &= \sum_{x \in H \backslash G} \langle f_{1,v}(x), f_{1,v}(x) \rangle_V \\ &= \langle f_{1,v}(1), f_{1,v}(1) \rangle_V = \langle v, v \rangle = 1, \end{aligned}$$

as needed. □

2. Cuspidal representations of $\mathrm{GL}_2(k)$

Let q be a prime power, let k be the finite field with q elements, let L be the unique quadratic extension of k, and let $G = \mathrm{GL}_2(k)$. Define the subgroups

$$U = \left\{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) \in G \right\}, \quad Z = \left\{ \left(\begin{smallmatrix} a & \\ & a \end{smallmatrix} \right) \in G \right\}, \quad B = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) \in G \right\}.$$

Note that Z is the center of G. Recall that the cuspidal representations of G are those that do not contain the trivial character of the unipotent subgroup U. These are precisely the irreducible representations that do not arise via parabolic induction. They have dimension $q-1$, and are parametrized by the Galois orbits of the primitive characters of L^*, defined below.

2.1. ***Primitive characters of L^*.*** For any finite abelian group H, we write $\widehat{H}$ for the dual group, consisting of the characters $\chi : H \longrightarrow \mathbb{C}^*$. We recall that

$$H \cong \widehat{H}.$$

Thus $\widehat{L^*}$ is a cyclic group of order $q^2 - 1$. A character $\nu \in \widehat{L^*}$ is *primitive* if $\nu^q \neq \nu$. Otherwise, ν is *imprimitive*. Letting $\bar{\alpha} = \alpha^q$ denote the Frobenius map, the norm map $N : L \longrightarrow k$ is given by

$$N(\alpha) = \alpha\bar{\alpha} = \alpha^{q+1}.$$

Proposition 2.1. *Let $\nu : L^* \to \mathbb{C}^*$ be a character of L^*. Then the following are equivalent*:

(i) *ν is imprimitive*; *that is*, $\nu^q = \nu$.

(ii) $\nu = \chi \circ N$ *for some* $\chi \in \widehat{k^*}$.

(iii) $L^1 \subset \ker(\nu)$, *where L^1 is the subgroup of norm* 1 *elements of* L^*.

Proof. Let θ be a generator of the cyclic group L^*. Then θ^{q+1} is a generator of k^*. If $\nu^q = \nu$, then $\nu(\theta)$ is a $(q-1)$-st root of unity, so we may define $\chi \in \widehat{k^*}$ by $\chi(\theta^{q+1}) = \nu(\theta)$. Then $\nu = \chi \circ N$, so (i) implies (ii). It is clear that (ii) implies (iii). On the other hand, for any $x \in L^*$, $N(x^{q-1}) = N(x)^{q-1} = 1$, so that $x^{q-1} \in L^1$. Therefore if (iii) holds, $\nu^q(x) = \nu(x^q) = \nu(x^{q-1}x) = \nu(x^{q-1})\nu(x) = \nu(x)$. Hence (iii) implies (i). □

The imprimitive characters thus correspond bijectively with the characters of k^*, so there are $q-1$ of them. It follows that there are $(q^2-1)-(q-1) = q^2-q$ primitive characters of L^*.

Lemma 2.2. *Let ν be a primitive character of L^*. Then, for all $\alpha \in k^*$,*

$$\sum_{\substack{x \in L^* \\ N(x)=\alpha}} \nu(x) = 0.$$

Proof. By Proposition 2.1, there exists $\lambda \in L^1$ such that $\nu(\lambda) \neq 1$. Thus,

$$\sum_{N(x)=\alpha} \nu(x) = \sum_{N(x)=\alpha} \nu(\lambda x) = \nu(\lambda) \sum_{N(x)=\alpha} \nu(x).$$

It follows that $\sum_{N(x)=\alpha} \nu(x) = 0$. □

Next, we examine the partition of primitive characters into classes according to their restrictions to k^*. This will allow us to count the number of depth-zero supercuspidal representations with a given central character (see (4-1)).

Proposition 2.3. *Suppose ω is a given character of k^*. Let P_ω denote the number of primitive characters ν of L^* for which $\nu|_{k^*} = \omega$. Then*

$$P_\omega = \begin{cases} q-1 & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is trivial}, \\ q+1 & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is nontrivial}, \\ q & \text{if } q \text{ is even}. \end{cases}$$

Proof. Let ξ be a generator of the cyclic group $\widehat{L^*}$. Let $\nu_0 = \xi^{q-1}$. Note that for $\alpha \in k^*$,

$$\nu_0(\alpha) = \xi(\alpha^{q-1}) = \xi(1) = 1.$$

In fact, ν_0 is a generator of the order $q+1$ subgroup $\widehat{L^*/k^*}$ of $\widehat{L^*}$. Let us consider two characters of L^* to be equivalent if they have the same restriction to k^*. Then the equivalence class of a given character ν is the set

$$\{\nu, \nu\nu_0, \nu\nu_0^2, \dots, \nu\nu_0^q\}. \tag{2-1}$$

If $\omega = \nu|_{k^*}$, then $P_\omega = q+1-A_\omega$, where A_ω is the number of imprimitive elements of the above set. Write $\nu = \xi^b$. Without loss of generality (replacing ν by some $\nu\nu_0^m$), we may assume that $0 \le b < q-1$. A character ξ^a is imprimitive if and only if $\xi^{qa} = \xi^a$, or equivalently, $(q+1) \mid a$. Suppose $\nu\nu_0^s$ and $\nu\nu_0^k$ are both imprimitive for $0 \le s \le k \le q$. Then $b+(q-1)s$ and $b+(q-1)k$ are both divisible by $q+1$ and strictly less than q^2-1. Their difference $(k-s)(q-1) \ge 0$ also has these properties, and furthermore it is divisible by

$$\operatorname{lcm}(q-1, q+1) = \begin{cases} (q^2-1)/2 & \text{if } q \text{ is odd,} \\ q^2-1 & \text{if } q \text{ is even.} \end{cases}$$

It follows that $k-s=0$ if q is even, and $k-s \in \{0, (q+1)/2\}$ if q is odd. This means that $A_\omega \le 1$ if q is even, and $A_\omega \le 2$ if q is odd.

Suppose q is odd and b is even. Then there are two imprimitive elements, namely

$$b + \tfrac{1}{2}b(q-1) = \tfrac{1}{2}b(q+1), \tag{2-2}$$

giving $\nu\nu_0^{b/2} = \xi^{b/2} \circ N$, and

$$b + \frac{b+q+1}{2}(q-1) = \frac{b+q-1}{2}(q+1), \tag{2-3}$$

giving $\nu\nu_0^{(b+q+1)/2} = \xi^{(b+q-1)/2} \circ N$. Hence $A_\omega = 2$ in this case. Noting that b is even if and only if $\omega^{(q-1)/2} = 1$, we obtain the first claim of the proposition: $P_\omega = q+1-A_\omega = q-1$.

Suppose q is odd and b is odd. Then for all k, $b+k(q-1)$ is odd, and hence it cannot be divisible by the even number $q+1$. So $A_\omega = 0$, and $P_\omega = q+1$, proving the second claim of the proposition.

If q is even, then as shown above, $A_\omega \le 1$. If b is even then (2-2) is a solution, and if b is odd, (2-3) is a solution. Either way, this shows that $A_\omega \ge 1$, and hence $A_\omega = 1$, proving the final claim that $P_\omega = q+1-A_\omega = q$ when q is even. □

The character sums in the next proposition will be used in Section 4 when we sum matrix coefficients over all representations with a given central character.

Proposition 2.4. *Let ω be a character of k^*, and let $[\omega]$ denote the set of primitive characters of L^* extending ω.*

Suppose q is odd and $\omega^{(q-1)/2}$ is nontrivial. Then for $\alpha \in L^*$,

$$\sum_{\nu\in[\omega]} \nu(\alpha) = \begin{cases} (q+1)\omega(\alpha) & \text{if } \alpha \in k^*, \\ 0 & \text{if } \alpha \notin k^*. \end{cases} \tag{2-4}$$

Suppose q is odd and $\omega^{(q-1)/2}$ is trivial. Then for $\alpha \in L^*$,

$$\sum_{\nu\in[\omega]} \nu(\alpha) = \begin{cases} (q-1)\omega(\alpha) & \text{if } \alpha \in k^*, \\ -2\omega(\alpha^{(q+1)/2}) & \text{if } \alpha \notin k^*, \alpha^{(q^2-1)/2} = 1, \\ 0 & \text{if } \alpha^{(q^2-1)/2} = -1. \end{cases} \tag{2-5}$$

(*Note that necessarily $\alpha \notin k^*$ if $\alpha^{(q^2-1)/2} \neq 1$.*)

Suppose q is even. Then for $\alpha \in L^*$,

$$\sum_{\nu\in[\omega]} \nu(\alpha) = \begin{cases} q\omega(\alpha) & \text{if } \alpha \in k^*, \\ -\omega(N(\alpha)^{1/2}) & \text{if } \alpha \notin k^*. \end{cases} \tag{2-6}$$

Here, we note that the square root is unique in k^*, since the square function is a bijection when q is even.

Proof. If $\alpha \in k^*$, then the sum is equal to $P_\omega \omega(\alpha)$ and the assertions follow from the previous proposition. So we may assume that $\alpha \notin k^*$. We use the notation from the previous proof. Suppose q is odd and $\omega^{(q-1)/2}$ is nontrivial. By the proof of the previous proposition,

$$\sum_{\nu\in[\omega]} \nu(\alpha) = \nu(\alpha) \sum_{m=0}^{q} \nu_0^m(\alpha),$$

where on the right-hand side ν is any fixed element of $[\omega]$. Noting that

$$\sum_{m=0}^{q} \nu_0^m(\alpha) = \sum_{\chi\in\widehat{L^*/k^*}} \chi(\alpha) = \begin{cases} q+1 & \text{if } \alpha \in k^*, \\ 0 & \text{if } \alpha \notin k^*, \end{cases} \tag{2-7}$$

(2-4) follows.

Now suppose q is odd and $\omega^{(q-1)/2}$ is trivial. By the proof of the previous proposition,

$$\sum_{\nu\in[\omega]} \nu(\alpha) = \xi^b(\alpha)\biggl(\sum_{m=0}^{q} \nu_0^m(\alpha) - \nu_0(\alpha)^{b/2} - \nu_0(\alpha)^{(b+q+1)/2}\biggr).$$

Since $\alpha \notin k^*$, by (2-7) this is equal to

$$-\xi^b(\alpha)\bigl[\nu_0(\alpha)^{b/2} + \nu_0(\alpha)^{b/2}\nu_0(\alpha)^{(q+1)/2}\bigr].$$

Recalling that $\nu_0 = \xi^{q-1}$ and writing $\xi^b = \nu$, this is

$$= -\nu\big(\alpha^{1+((q-1)/2)}\big)\big[1+\xi\big(\alpha^{(q^2-1)/2}\big)\big] = -\nu\big(\alpha^{(q+1)/2}\big)\big[1+\xi(\alpha^{(q^2-1)/2})\big]. \quad (2\text{-}8)$$

Observe that $\alpha^{(q^2-1)/2} = \pm 1$ since its square is 1. If it is equal to $+1$, then $\alpha^{(q+1)/2} \in k^*$ since its $(q-1)$-st power is 1, and we immediately obtain the middle line of (2-5). Otherwise $\xi\big(\alpha^{(q^2-1)/2}\big) = \xi(-1) = -1$ and (2-8) vanishes.

When q is even, there exists a choice of ξ for which $\omega = \xi^b$ with b even. Then using (2-2), we find that when $\alpha \notin k^*$,

$$\sum_{\nu\in[\omega]} \nu(\alpha) = -\xi^{b/2}(N(\alpha)) = -\omega\big(N(\alpha)^{1/2}\big). \qquad \square$$

2.2. ***Model for cuspidal representations.*** There are various ways to construct the cuspidal representation ρ_ν attached to a primitive character ν. The action of L^* on the k-vector space $L \cong k^2$ by multiplication gives an identification

$$L^* \cong T \qquad (2\text{-}9)$$

of L^* with a nonsplit torus $T \subset G$, with $k^* \subset L^*$ mapping onto $Z \subset T$. The characteristic polynomial of an element $g \in G$ is irreducible over k if and only if g is conjugate to an element of $T - Z$.

Fix a nontrivial character of the additive group

$$\psi : k \longrightarrow \mathbb{C}^*,$$

viewed in the obvious way as a character of U. Then one may define ρ_ν implicitly by

$$\mathrm{Ind}_{ZU}^G(\nu \otimes \psi) = \rho_\nu \oplus \mathrm{Ind}_T^G\, \nu; \qquad (2\text{-}10)$$

see [Bushnell and Henniart 2006, Theorem 6.4]. Although (2-10) allows for computation of the trace of ρ_ν (see (2-20) below), it is not convenient for computing the matrix coefficients. For this purpose we shall use the explicit model for ρ_ν defined in [Piatetski-Shapiro 1983, Section 13] as follows.[1]

Given a primitive character ν of L^* and ψ as above, let

$$V = \mathbb{C}[k^*]$$

[1] There is a minus sign missing from the definition of $j(x)$ in Equation (4) of Section 13 of [Piatetski-Shapiro 1983] (otherwise his identity (6) will not hold). Likewise a minus sign is missing from (16) on page 40. The expression four lines above (16) is correct (except K should be K^*).

be the vector space of functions $f : k^* \to \mathbb{C}$. We define a representation ρ_ν of G on V as follows. For any $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B$, $f \in V$, let

$$\left[\rho_\nu \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} f\right](x) = \nu(d)\psi(bd^{-1}x) f(ad^{-1}x) \quad (x \in k^*), \tag{2-11}$$

and for $g \in G - B$, define

$$(\rho_\nu(g)f)(x) = \sum_{y \in k^*} \phi(x, y; g) f(y) \quad (x \in k^*), \tag{2-12}$$

where, for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G - B$,

$$\phi(x, y; g) = -\frac{1}{q}\psi\left[\frac{ax+dy}{c}\right] \sum_{\substack{t \in L^* \\ N(t) = xy^{-1}\det g}} \psi\left(-\frac{y}{c}(t+\bar{t})\right)\nu(t). \tag{2-13}$$

Theorem 2.5. *If ν is a primitive character of L^*,* (2-11) *and* (2-12) *give a well-defined representation (ρ_ν, V) which is cuspidal. Furthermore, every cuspidal representation is isomorphic to some ρ_ν, and $\rho_\nu \cong \rho_{\nu'}$ if and only if $\nu' \in \{\nu, \nu^q\}$. In particular, there are $(q^2 - q)/2$ distinct cuspidal representations.*

Proof. See [Piatetski-Shapiro 1983, Section 13–14], where it is assumed that $q > 2$ throughout. When $q = 2$, G is isomorphic to the symmetric group S_3. The unique cuspidal representation is the character sending each permutation to its sign. It is readily checked that the above construction defines this character as well, so the theorem remains valid when $q = 2$. □

Define an inner product on V by

$$\langle f_1, f_2\rangle = \sum_{x \in k^*} f_1(x)\overline{f_2(x)}. \tag{2-14}$$

We will work with the orthonormal basis

$$\mathcal{B} = \{f_r\}_{r \in k^*} \quad \text{for } f_r(x) = \begin{cases} 1 & \text{if } x = r, \\ 0 & \text{if } x \neq r. \end{cases}$$

Proposition 2.6. *Let ν be a primitive character of L^*, and let ρ_ν be the associated cuspidal representation of G. Then ρ_ν is unitary with respect to the inner product* (2-14).

Proof. By linearity, it suffices to prove that for all $f_r, f_s \in \mathcal{B}$ and $g \in G$,

$$\langle \rho_\nu(g) f_r, \rho_\nu(g) f_s\rangle = \langle f_r, f_s\rangle. \tag{2-15}$$

By the Bruhat decomposition $G = B \cup Bw'U$ for $w' = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ and the fact that ρ_ν is a homomorphism, we only need to consider $g \in B$ and $g = w'$.

Suppose first that $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B$. Then by (2-11),

$$\left\langle \rho_\nu(g) f_r, \rho_\nu(g) f_s \right\rangle = \sum_{x\in k^*} \nu(d)\psi(bd^{-1}x) f_r(ad^{-1}x)\overline{\nu(d)\psi(bd^{-1}x) f_s(ad^{-1}x)}.$$

Using the fact that ν and ψ are unitary, and replacing x by $a^{-1}dx$, we see that this expression equals

$$\sum_{x\in k^*} f_r(x)\overline{f_s(x)} = \langle f_r, f_s\rangle,$$

as needed.

It remains to prove (2-15) for $g = w' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. By (2-12),

$$\begin{aligned}\rho_\nu(w') f(x) &= \sum_{y\in k^*} \phi(x, y; w') f(y) = -\frac{1}{q} \sum_{y\in k^*} \sum_{\substack{t\in L^* \\ N(t)=xy^{-1}}} \psi(y(t+\bar{t}))\nu(t) f(y) \\ &= -\frac{1}{q} \sum_{y\in k^*} \sum_{\substack{u\in L^* \\ N(u)=xy}} \psi(u+\bar{u})\nu(u)\nu(y^{-1}) f(y) = \sum_{y\in k^*} \nu(y^{-1}) j(xy) f(y),\end{aligned}$$

where

$$j(t) = -\frac{1}{q} \sum_{\substack{u\in L^* \\ N(u)=t}} \psi(u+\bar{u})\nu(u). \tag{2-16}$$

Hence

$$\rho_\nu(w') f_r(x) = \sum_{y\in k^*} \nu(y^{-1}) j(xy) f_r(y) = \nu(r^{-1}) j(rx).$$

We now see that

$$\begin{aligned}\langle \rho_\nu(w') f_r, \rho_\nu(w') f_s\rangle &= \sum_{x\in k^*} \nu(r^{-1}) j(rx)\overline{\nu(s^{-1}) j(sx)} = \nu(sr^{-1}) \sum_{x\in k^*} j(rx)\overline{j(sx)} \\ &= \nu(sr^{-1}) \sum_{x\in k^*} j(rs^{-1}x)\overline{j(x)}.\end{aligned}$$

Taking $r' = rs^{-1}$, it suffices to prove that

$$\sum_{x\in k^*} j(r'x)\overline{j(x)} = \begin{cases} 0 & \text{if } r' \neq 1, \\ 1 & \text{if } r' = 1. \end{cases} \tag{2-17}$$

From the definition (2-16), we have

$$\begin{aligned}\sum_{x\in k^*} j(r'x)\overline{j(x)} &= \frac{1}{q^2} \sum_{x\in k^*} \sum_{N(\alpha)=r'x} \sum_{N(\beta)=x} \psi(\alpha+\bar{\alpha})\nu(\alpha)\psi(-\beta-\bar{\beta})\nu(\beta^{-1}) \\ &= \frac{1}{q^2} \sum_{\beta\in L^*} \sum_{N(\alpha)=r'N(\beta)} \psi(\alpha+\bar{\alpha}-\beta-\bar{\beta})\nu(\alpha\beta^{-1}).\end{aligned}$$

Since the norm map is surjective, there exists $z \in L^*$ such that $N(z) = r'$. Then $\alpha = z\beta u$ for some $u \in L^1$. This allows us to rewrite the above sum as

$$\sum_{x\in k^*} j(r'x)\overline{j(x)} = \frac{1}{q^2}\sum_{\beta\in L^*}\sum_{u\in L^1}\psi\bigl(z\beta u + \overline{z\beta u} - \beta - \bar{\beta}\bigr)\nu(zu)$$
$$= \frac{\nu(z)}{q^2}\sum_{u\in L^1}\nu(u)\sum_{\beta\in L^*}\psi\bigl(\mathrm{tr}[(zu-1)\beta]\bigr). \tag{2-18}$$

Generally, for $c \in L$, the map $R(\beta) = \psi(\mathrm{tr}[c\beta])$ is a homomorphism from L to $\mathbb{C}^*$. If $c \neq 0$, then R is nontrivial since the trace map from L to k is surjective. It follows that

$$\sum_{\beta\in L^*}\psi\bigl(\mathrm{tr}[c\beta]\bigr) = \begin{cases} -1 & \text{if } c \neq 0, \\ q^2 - 1 & \text{if } c = 0. \end{cases} \tag{2-19}$$

Suppose $r' \neq 1$. Then $N(zu) = N(z) = r' \neq 1$, so in particular $zu \neq 1$. Therefore (2-18) becomes

$$\sum_{x\in k^*} j(r'x)\overline{j(x)} = -\frac{\nu(z)}{q^2}\sum_{u\in L^1}\nu(u) = 0,$$

where we have used the fact (Proposition 2.1) that ν is a nontrivial character of L^1 since ν is primitive.

Now suppose $r' = 1$. Then we can take $z = 1$, so by (2-18) and (2-19),

$$\sum_{x\in k^*} j(x)\overline{j(x)} = \frac{1}{q^2}\sum_{u\in L^1}\nu(u)\sum_{\beta\in L^*}\psi\bigl(\mathrm{tr}[(u-1)\beta]\bigr)$$
$$= \frac{q^2-1}{q^2} - \frac{1}{q^2}\sum_{\substack{u\in L^1\\ u\neq 1}}\nu(u) = \left(1 - \frac{1}{q^2}\right) + \frac{1}{q^2} = 1,$$

since $\sum_{\substack{u\in L^1\\ u\neq 1}}\nu(u) = -1$, again because ν is nontrivial on L^1. This proves (2-17). □

2.3. ***Matrix coefficients of cuspidal representations.*** Let ν be a primitive character of L^*. Using (2-10), one finds that

$$\mathrm{tr}\,\rho_\nu(x) = \begin{cases} (q-1)\nu(x) & \text{if } x \in Z, \\ -\nu(z) & \text{if } x = zu,\ z \in Z,\ u \in U,\ u \neq 1, \\ -\nu(x) - \nu^q(x) & \text{if } x \in T,\ x \notin Z, \\ 0 & \text{if no conjugate of } x \text{ is in } T \cup ZU \end{cases} \tag{2-20}$$

(see [Bushnell and Henniart 2006, (6.4.1)]). This is a sum of matrix coefficients. For the coefficients themselves, we use the model given in the previous section to prove the following (which can also be used to derive (2-20)).

Theorem 2.7. *Let* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ *and* $f_r, f_s \in \mathcal{B}$. *Let* ρ_ν *be a cuspidal representation of* G. *If* $g \in B$, *then*

$$\langle \rho_\nu(g) f_r, f_s \rangle = \begin{cases} \nu(d)\psi(bd^{-1}s) & \text{if } r = ad^{-1}s, \\ 0 & \text{if } r \neq ad^{-1}s. \end{cases}$$

If $g \notin B$, *then*

$$\langle \rho_\nu(g) f_r, f_s \rangle = \phi(s, r; g),$$

where ϕ *is defined in* (2-13).

Proof. First suppose $g \in B$ (i.e., $c = 0$). Then

$$\begin{aligned} \langle \rho_\nu(g) f_r, f_s \rangle &= \sum_{x \in k^*} [\rho_\nu(g) f_r](x) \overline{f_s(x)} = [\rho_\nu(g) f_r](s) \\ &= \nu(d)\psi(bd^{-1}s) f_r(ad^{-1}s) = \begin{cases} \nu(d)\psi(bd^{-1}s) & \text{if } r = ad^{-1}s, \\ 0 & \text{if } r \neq ad^{-1}s. \end{cases} \end{aligned}$$

Now, suppose $g \notin B$. Then

$$[\rho_\nu(g) f_r](x) = \sum_{y \in k^*} \phi(x, y; g) f_r(y) = \phi(x, r; g).$$

Therefore

$$\langle \rho_\nu f_r, f_s \rangle = \sum_{x \in k^*} [\rho_\nu(g) f_r](x) \overline{f_s(x)} = [\rho_\nu(g) f_r](s) = \phi(s, r; g),$$

as needed. □

3. Depth-zero supercuspidal representations of $\mathrm{GL}_2(F)$

We move now to the $\mathfrak{p}$-adic setting. When no field is specified, G, Z, B, U, etc., will henceforth denote the corresponding subgroups of $\mathrm{GL}_2(F)$ rather than $\mathrm{GL}_2(k)$. Fix a primitive character ν, and let ρ_ν be the associated cuspidal representation of $\mathrm{GL}_2(k)$. We view ρ_ν as a representation of $K = \mathrm{GL}_2(\mathfrak{o})$ via reduction modulo $\mathfrak{p}$:

$$K \longrightarrow \mathrm{GL}_2(k) \longrightarrow \mathrm{GL}(V).$$

The central character of this representation is given by $z \mapsto \nu(z(1+\mathfrak{p}))$ for $z \in \mathfrak{o}^*$. Extend this character of $\mathfrak{o}^*$ to $Z \cong F^* = \bigcup_{n \in \mathbb{Z}} \varpi^n \mathfrak{o}^*$ by choosing a complex number $\nu(\varpi)$ of absolute value 1. We denote this character of F^* by ν. This allows us to view ρ_ν as a unitary representation of the group ZK. Let

$$\pi_\nu = \text{c-Ind}_{ZK}^{\mathrm{GL}_2(F)}(\rho_\nu)$$

be the representation of G compactly induced from ρ_ν. This representation is irreducible and supercuspidal (see, for example, [Bump 1997, Theorem 4.8.1]).

3.1. ***Matrix coefficients.*** Define an inner product on $\text{c-Ind}_{ZK}^{G}(\rho_\nu)$ by

$$\langle f_1, f_2\rangle = \sum_{x\in ZK\backslash G} \big\langle f_1(x), f_2(x)\big\rangle_V, \tag{3-1}$$

where $\langle\,\cdot\,,\cdot\,\rangle_V$ denotes the inner product on V defined in (2-14). As in Section 1, this inner product is G-equivariant.

The matrix coefficients of π_ν can now be computed explicitly by using Proposition 1.1 in conjunction with Theorem 2.7. Likewise, by Proposition 1.2, if we normalize so that $\text{meas}(\overline{K}) = 1$, then

$$d_{\pi_\nu} = \dim\rho_\nu = q-1. \tag{3-2}$$

Define a function $\phi_\nu : G \longrightarrow \mathbb{C}$ by

$$\phi_\nu(x) = \begin{cases} \overline{\operatorname{tr}\rho_\nu(x)} & \text{if } x\in ZK,\\ 0 & \text{otherwise.}\end{cases} \tag{3-3}$$

Then this is a *pseudocoefficient* of π_ν in the sense that for any irreducible tempered representation π of G with central character ω,

$$\operatorname{tr}\pi(\phi_\nu) = \begin{cases} 1 & \text{if } \pi\cong\pi_\nu,\\ 0 & \text{otherwise}\end{cases}$$

(see [Palm 2012, Section 9.4.1]). The function ϕ_ν may be computed explicitly using (2-20).

3.2. ***New vectors.*** For an integer $n \geq 0$, define the congruence subgroup

$$K_1(\mathfrak{p}^n) = \big\{\big(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\big)\in K \mid c, (d-1)\in\mathfrak{p}^n\big\}.$$

If π is a representation of G, we let $\pi^{K_1(\mathfrak{p}^n)}$ denote the space of vectors fixed by $K_1(\mathfrak{p}^n)$. By a result of Casselman [1973], for any irreducible admissible representation π of G, there exists a unique ideal $\mathfrak{p}^n$ (the *conductor* of π) for which $\dim\pi^{K_1(\mathfrak{p}^n)} = 1$ and $\dim\pi^{K_1(\mathfrak{p}^{n-1})} = 0$. A nonzero vector fixed by $K_1(\mathfrak{p}^n)$ is called a *new vector*. The supercuspidal representations constructed above have conductor $\mathfrak{p}^2$. We shall give an elementary proof below, and exhibit a new vector. More generally, the new vectors for depth-zero supercuspidal representations of $\mathrm{GL}_n(F)$ were identified by Reeder [1991, Example (2.3)].

Proposition 3.1. *The supercuspidal representation π_ν defined above has conductor $\mathfrak{p}^2$. If we let $w\in\mathbb{C}[k^*]$ denote the constant function* 1, *that is*,

$$w = \sum_{r\in k^*} f_r, \tag{3-4}$$

then the function $f = f_{\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right),w}$ *supported on the coset*

$$ZK\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix} = ZK\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}K_1(\mathfrak{p}^2),$$

and defined by

$$f\left(zk\begin{pmatrix}\varpi & 0\\ 0 & 1\end{pmatrix}\right) = \rho_\nu(zk)w,$$

is a new vector of π_ν.

Proof. To see that f is $K_1(\mathfrak{p}^2)$-invariant, it suffices to show that

$$f\left(\left(\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right)k\right) = w \quad \text{for all } k \in K_1(\mathfrak{p}^2).$$

Writing $k = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ with $c \in \mathfrak{p}^2$ and $d \in 1+\mathfrak{p}^2$, we have

$$\begin{aligned} f\left(\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}k\right) &= f\left(\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\varpi^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix}\varpi & \\ & 1\end{pmatrix}\right)\\ &= \rho_\nu\left(\begin{pmatrix}a & \varpi b\\ \varpi^{-1}c & d\end{pmatrix}\right)w = \rho_\nu\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)w\\ &= \sum_{r\in k^*}\rho_\nu\left(\begin{pmatrix}a & \\ & 1\end{pmatrix}\right)f_r = \sum_{r\in k^*} f_{a^{-1}r} = w, \end{aligned}$$

as needed. This shows that $\pi_\nu^{K_1(\mathfrak{p}^2)} \neq 0$, so the conductor divides $\mathfrak{p}^2$. There are various ways to see that the conductor is exactly $\mathfrak{p}^2$. When $n > 1$, it is straightforward to show that a continuous irreducible n-dimensional complex representation of the Weil group of F has Artin conductor of exponent at least n (see, for example, [Gross and Reeder 2010, Equation (1)]). So by the local Langlands correspondence, the conductor of any supercuspidal representation of $\mathrm{GL}_n(F)$ is divisible by $\mathfrak{p}^n$, giving the desired conclusion here when $n = 2$. For an elementary proof in the present situation, one can observe that a function $f \in \text{c-Ind}_{ZK}^G(\rho_\nu)$ supported on a coset ZKx is $K_1(\mathfrak{p})$-invariant if and only if ρ_ν is trivial on $K \cap xK_1(\mathfrak{p})x^{-1}$. Using the double coset decomposition

$$G = \bigcup_{n\geq 0} ZK\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}K = \bigcup_{n\geq 0}\ \bigcup_{\delta\in \overline{K}/\overline{K_1(\mathfrak{p})}} ZK\begin{pmatrix}\varpi^n & \\ & 1\end{pmatrix}\delta K_1(\mathfrak{p})$$

(we may use the representatives $\delta \in \{1\} \cup \left\{\left(\begin{smallmatrix}y & 1\\ 1 & 0\end{smallmatrix}\right) \,\middle|\, y \in \mathfrak{o}/\mathfrak{p}\right\}$; see, for example, [Knightly and Li 2006, Lemma 13.1]), it suffices to consider $x = \left(\begin{smallmatrix}\varpi^n & \\ & 1\end{smallmatrix}\right)\delta$, and one checks that in each case ρ_ν is *not* trivial on $K \cap xK_1(\mathfrak{p})x^{-1}$, so $\pi_\nu^{K_1(\mathfrak{p})} = \{0\}$. □

3.3. ***Matrix coefficient of the new vector.*** Generally, if π is a supercuspidal representation with unit new vector v and formal degree d_π, the function

$$g \mapsto d_\pi \overline{\langle \pi(g)v, v\rangle}$$

can be used to define a projection operator onto $\mathbb{C}v$.

In the present context, if f is the new vector defined in Proposition 3.1, one finds easily that $\|f\|^2 = (q-1)$, so with the standard normalization $\operatorname{meas}(\overline{K}) = 1$, by (3-2) we have

$$\Phi_\nu(g) \stackrel{\text{def}}{=} d_{\pi_\nu} \overline{\left\langle \pi_\nu(g)\frac{f}{\|f\|}, \frac{f}{\|f\|}\right\rangle} = \overline{\langle \pi_\nu(g)f, f\rangle}. \tag{3-5}$$

By Proposition 1.1,

$$\operatorname{Supp}(\Phi_\nu) = \begin{pmatrix} \varpi^{-1} & \\ & 1\end{pmatrix} ZK \begin{pmatrix} \varpi & \\ & 1\end{pmatrix},$$

and for $g = \left(\begin{smallmatrix} \varpi^{-1} & \\ & 1\end{smallmatrix}\right) h \left(\begin{smallmatrix} \varpi & \\ & 1\end{smallmatrix}\right) \in \operatorname{Supp}(\Phi_\nu)$,

$$\Phi_\nu(g) = \overline{\langle \rho_\nu(h)w, w\rangle}_V \tag{3-6}$$

for w as in (3-4). This is computed as follows.

Theorem 3.2. *Let* $h = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in G(k) = \mathrm{GL}_2(k)$, *and let* $w \in V$ *be the function defined in* (3-4). *Then*

$$\langle \rho_\nu(h)w, w\rangle_V = \begin{cases} (q-1)\nu(d) & \text{if } b = c = 0, \\ -\nu(d) & \text{if } c = 0,\ b \neq 0, \\ -\displaystyle\sum_{\substack{\alpha \in L^* \\ \alpha + \bar{\alpha} = \frac{aN(\alpha)}{\det h} + d}} \nu(\alpha) & \text{if } c \neq 0. \end{cases}$$

Remark. The sum may be evaluated using Proposition 3.3 below.

Proof. To ease notation, we drop the subscript V from the inner product. Suppose $h = \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right) \in B(k)$. Then applying Theorem 2.7,

$$\langle \rho_\nu(h)w, w\rangle = \sum_{r,s \in k^*} \langle \rho_\nu(h) f_r, f_s\rangle = \sum_{s \in k^*} \langle \rho_\nu(h) f_{ad^{-1}s}, f_s\rangle = \nu(d) \sum_{s \in k^*} \psi(bd^{-1}s).$$

This gives

$$\langle \rho_\nu(h)w, w\rangle = \begin{cases} (q-1)\nu(d) & \text{if } b = 0, \\ -\nu(d) & \text{if } b \neq 0. \end{cases}$$

Now suppose $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(k) - B(k)$. Then by Theorem 2.7,

$$\begin{aligned}\langle \rho_\nu(h)w, w\rangle &= \sum_{r,s\in k^*} \phi(s, r; h)\\ &= -\frac{1}{q}\sum_{r,s\in k^*} \psi\big(c^{-1}(sa+rd)\big) \sum_{\substack{\alpha\in L^* \\ N(\alpha)=sr^{-1}\det h}} \psi\big(-rc^{-1}(\alpha+\bar{\alpha})\big)\nu(\alpha).\end{aligned}$$

Let $l = sr^{-1}$, so $s = rl$. From the previous display we have

$$\begin{aligned}\langle \rho_\nu(h)w, w\rangle &= -\frac{1}{q}\sum_{r\in k^*}\sum_{l\in k^*} \psi\big(c^{-1}(rla+rd)\big) \sum_{N(\alpha)=l\det h} \psi\big(-rc^{-1}(\alpha+\bar{\alpha})\big)\nu(\alpha)\\ &= -\frac{1}{q}\sum_{l\in k^*}\sum_{N(\alpha)=l\det h} \nu(\alpha)\sum_{r\in k^*}\psi\big(rc^{-1}(al+d-(\alpha+\bar{\alpha}))\big)\\ &= -\frac{1}{q}\sum_{l\in k^*}\sum_{N(\alpha)=l\det h} \nu(\alpha)\sum_{r\in k^*}\psi\big(r(al+d-(\alpha+\bar{\alpha}))\big).\end{aligned}$$

There are two cases for the inner sum:

$$\sum_{r\in k^*}\psi\big(r(al+d-(\alpha+\bar{\alpha}))\big) = \begin{cases} -1 & \text{if } al+d-(\alpha+\bar{\alpha}) \neq 0,\\ q-1 & \text{if } al+d-(\alpha+\bar{\alpha}) = 0.\end{cases}$$

Therefore,

$$\begin{aligned}\big\langle \rho_\nu(h)w, w\big\rangle &= -\frac{1}{q}\sum_{l\in k^*}\Bigg(\sum_{\substack{N(\alpha)=l\det h \\ \alpha+\bar{\alpha}\neq al+d}} -\nu(\alpha) + \sum_{\substack{N(\alpha)=l\det h \\ \alpha+\bar{\alpha}=al+d}} (q-1)\nu(\alpha)\Bigg)\\ &= -\frac{1}{q}\sum_{l\in k^*}\Bigg(-\sum_{N(\alpha)=l\det h}\nu(\alpha) + q\sum_{\substack{N(\alpha)=l\det h \\ \alpha+\bar{\alpha}=al+d}}\nu(\alpha)\Bigg).\end{aligned}$$

By Lemma 2.2, the first sum in the big parentheses vanishes. So

$$\big\langle \rho_\nu(h)w, w\big\rangle = -\frac{1}{q}\sum_{l\in k^*} q \sum_{\substack{N(\alpha)=l\det h \\ \alpha+\bar{\alpha}=al+d}} \nu(\alpha) = -\sum_{\substack{\alpha\in L^* \\ \alpha+\bar{\alpha}=\frac{aN(\alpha)}{\det h}+d}} \nu(\alpha), \tag{3-7}$$

as claimed. □

This sum can be refined as follows.

Proposition 3.3. *For $l \in k^*$ and $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(k)$ with $c \neq 0$, define*

$$p_{h,l}(X) = X^2 - \left(\frac{al}{\det h} + d\right)X + l \in k[X]. \tag{3-8}$$

Then

$$\langle \rho_\nu(h)w, w\rangle = -\sum_{\substack{l\in k^* \\ p_{h,l}(X)\text{ irred. over } k}} \sum_{\substack{\text{roots }\alpha\text{ of} \\ p_{h,l}(X)\text{ in } L^*}} \nu(\alpha) - \sum_{\substack{l\in k^* \\ p_{h,l}(X)=(X-\alpha)^2 \text{in } k[X]}} \nu(\alpha). \quad (3\text{-}9)$$

Proof. If $\alpha \in L^*$ contributes to the sum (3-7), then it is a root of

$$X^2 - \left(\frac{aN(\alpha)}{\det h} + d\right)X + N(\alpha).$$

Conversely, given $l \in k^*$, a root α_l of $p_{h,l}(X)$ contributes to (3-7) if and only if $p_{h,l}(X) = (X-\alpha_l)(X-\bar{\alpha}_l)$ in $L[X]$, which is the case if and only if either $p_{h,l}(X)$ is irreducible or $p_{h,l}(X) = (X-\alpha_l)^2$ with $\alpha_l \in k^*$. The proposition now follows. □

3.4. ***Motivation.*** Although specific global applications of the above formulas are beyond the scope of this article, perhaps a few words of motivation will be helpful. The two functions ϕ_ν and Φ_ν of (3-3) and (3-5) serve slightly different purposes. The former is simpler and hence easier to work with. Taken as a local component of a global test function, it is well suited for use in the Arthur–Selberg trace formula, for example if one is interested in detecting those automorphic representations $\pi = \bigotimes_w \pi_w$ of the adelic group $\mathrm{GL}_2(\mathbf{A}_\mathbb{Q})$ with the local condition $\pi_w \cong \pi_\nu$ at a given finite place w (e.g., to obtain a dimension formula for the associated space of classical newforms). This method was treated in detail recently by Palm [2012]. On the other hand, as mentioned in the Introduction, the matrix coefficient Φ_ν gives rise to an operator projecting onto the span of the newforms attached to the global representations π as above (see [Knightly and Li 2012, Section 2.5]). It can be used in variants of the trace formula to extract finer information, like Fourier coefficients or L-values of these newforms. Of course, the utility of Φ_ν in explicit computation is limited by the complexity of the sum in Theorem 3.2.

4. Consideration of central character

The supercuspidal representations which have a given central character ω occur naturally together as irreducible subrepresentations of the right regular representation of $G(F)$ on the space of L^2 functions $f : G(F) \to \mathbb{C}$ that transform under the center by $\bar{\omega}$. It is often the case in number theory that one can achieve a certain amount of simplification by simultaneously treating all objects in a family via averaging. Here we sum the trace functions ϕ_ν from (3-3) over all isomorphism classes of depth-zero supercuspidal π_ν with a given central character, and similarly for the new vector matrix coefficients Φ_ν of (3-5).

Let ω be a unitary character of F^*, and let S_ω denote the set of isomorphism classes of depth-zero supercuspidal representations of $\mathrm{GL}_2(F)$ with central character

ω. In order for S_ω to be nonempty, ω must be trivial on $1+\mathfrak{p}$. In fact we have

$$|S_\omega| = \begin{cases} P_\omega/2 & \text{if } \omega\,|_{(1+\mathfrak{p})} = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{4-1}$$

for P_ω as in Proposition 2.3. (Note that ν and ν^q have the same restriction to k^* and $\rho_\nu \cong \rho_{\nu^q}$.)

Assuming $\omega\,|_{(1+\mathfrak{p})}$ is trivial, consider the sum of the trace functions ϕ_ν defined in (3-3). In view of the fact that $\phi_\nu = \phi_{\nu^q}$, we define

$$\phi_\omega = \frac{1}{2} \sum_{\nu \in [\omega]} \phi_\nu, \tag{4-2}$$

with notation as in Proposition 2.4. We can make it explicit with the following.

Theorem 4.1. *Suppose q is odd and $\omega^{(q-1)/2}$ is nontrivial. Then for $x \in G(k)$,*

$$\sum_{\nu\in[\omega]} \operatorname{tr} \rho_\nu(x) = \begin{cases} (q^2-1)\omega(x) & \textit{if } x \in Z, \\ -(q+1)\omega(z) & \textit{if } x = zu,\ z \in Z,\ u \in U,\ u \neq 1, \\ 0 & \textit{if no conjugate of } x \textit{ is in } ZU. \end{cases}$$

Suppose q is odd and $\omega^{(q-1)/2}$ is trivial. Then with T as in (2-9),

$$\sum_{\nu\in[\omega]} \operatorname{tr} \rho_\nu(x) = \begin{cases} (q-1)^2\omega(x) & \textit{if } x \in Z, \\ -(q-1)\omega(z) & \textit{if } x = zu,\ z \in Z,\ u \in U,\ u \neq 1, \\ 4\omega(x^{(q+1)/2}) & \textit{if } x \in T - Z,\ x^{(q^2-1)/2} = 1, \\ 0 & \textit{if } x \in T - Z,\ x^{(q^2-1)/2} = -1, \textit{ or if} \\ & \textit{no conjugate of } x \textit{ is in } T \cup ZU. \end{cases}$$

Suppose q is even. Then

$$\sum_{\nu\in[\omega]} \operatorname{tr} \rho_\nu(x) = \begin{cases} q(q-1)\omega(x) & \textit{if } x \in Z, \\ -q\omega(z) & \textit{if } x = zu,\ z \in Z,\ u \in U,\ u \neq 1, \\ 2\omega(N(x)^{1/2}) & \textit{if } x \in T - Z, \\ 0 & \textit{if no conjugate of } x \textit{ is in } T \cup ZU. \end{cases}$$

Proof. This follows immediately by examining the various cases using (2-20) and Proposition 2.4. □

Likewise, for a depth-zero supercuspidal representation $\pi = \pi_\nu$, let $\Phi_\pi = \Phi_\nu$ be the matrix coefficient defined in (3-5). Define a function Φ_ω on G by

$$\Phi_\omega(g) = \sum_{\pi \in S_\omega} \Phi_\pi(g) = \frac{1}{2} \sum_{\nu\in[\omega]} \overline{\langle \rho_\nu(h)w, w\rangle_V} \tag{4-3}$$

for $g = \left(\begin{smallmatrix} \varpi^{-1} & \\ & 1 \end{smallmatrix}\right) h \left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)$ with $h \in ZK$. In principle, this function can be used to define an operator that projects the automorphic spectrum of $\mathrm{GL}_2(\mathbf{A}_\mathbb{Q})$ onto the span of those newforms of a given weight and level p^2 that correspond to automorphic representations which are unramified away from p and are supercuspidal (as opposed to special or principal series) at p.

One can evaluate Φ_ω via the following.

Theorem 4.2. *Suppose $h = \left(\begin{smallmatrix} a & \\ & d \end{smallmatrix}\right) \in G(k)$ is diagonal. Then*

$$\sum_{\nu\in[\omega]} \langle \rho_\nu(h)w, w\rangle_V = \begin{cases} (q^2-1)\omega(d) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is nontrivial,} \\ (q-1)^2\omega(d) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is trivial,} \\ q(q-1)\omega(d) & \text{if } q \text{ is even.} \end{cases} \tag{4-4}$$

If $h = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B(k)$ with $b \neq 0$, then

$$\sum_{\nu\in[\omega]} \langle \rho_\nu(h)w, w\rangle_V = \begin{cases} -(q+1)\omega(d) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is nontrivial,} \\ -(q-1)\omega(d) & \text{if } q \text{ is odd and } \omega^{(q-1)/2} \text{ is trivial,} \\ -q\omega(d) & \text{if } q \text{ is even.} \end{cases} \tag{4-5}$$

If $g \in G(k) - B(k)$, then the sum is given by (4-6) *below.*

Proof. Suppose $h = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ is diagonal. Then by Theorem 3.2, we can write

$$\sum_{\nu\in[\omega]} \langle \rho_\nu(h)w, w\rangle = (q-1) \sum_{\nu\in[\omega]} \nu(d).$$

Applying Proposition 2.4 now gives (4-4), using the fact that $d \in k^*$.

Similarly, if $h = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $b \neq 0$, then applying Theorem 3.2,

$$\sum_{\nu\in[\omega]} \langle \rho_\nu(h)w, w\rangle = -\sum_{\nu\in[\omega]} \nu(d).$$

Using Proposition 2.4, this gives (4-5).

Now suppose $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(k) - B(k)$. By Proposition 3.3,

$$\sum_{\nu\in[\omega]} \langle \rho_\nu(h)w, w\rangle = -\sum_{\substack{l\in k^* \\ p_{h,l}(X) \text{ irred.}}} \sum_{\substack{\text{roots } \alpha \text{ of} \\ p_{h,l}(X) \text{ in } L^*}} \sum_{\nu\in[\omega]} \nu(\alpha) - \sum_{\substack{l\in k^* \\ p_{h,l}(X)=(X-\alpha)^2}} \sum_{\nu\in[\omega]} \nu(\alpha),$$

where $p_{h,l}(X) = X^2 - \big((al/\det h) + d\big)X + l \in k[X]$. If we fix one root $\alpha_l \in L^*$ of $p_{h,l}(X)$ for each l, then the above is

$$= -2 \sum_{\substack{l\in k^* \\ p_{h,l}(X) \text{ irred.}}} \sum_{\nu\in[\omega]} \nu(\alpha_l) - P_\omega \sum_{\substack{l\in k^* \\ p_{h,l}(X)=(X-\alpha_l)^2}} \omega(\alpha_l), \tag{4-6}$$

where $P_\omega = \big|[\omega]\big|$ as in Proposition 2.3. We have used the fact that

$$\sum_{\nu\in[\omega]} \nu(\alpha_l) = \sum_{\nu\in[\omega]} \nu(\bar{\alpha}_l),$$

since ν and ν^q both belong to $[\omega]$. Once again, (4-6) can be evaluated on a case-by-case basis using Proposition 2.4.

For instance, suppose q is odd. If $\omega^{(q-1)/2}$ is nontrivial, the first term of (4-6) vanishes. If $(al + d \det h) = 0$, then l makes no contribution to the second term. In particular, the term vanishes if $a = d = 0$. Generally, at most two l contribute to the second term when q is odd since $2\alpha_l = ((al/\det h) + d)$ implies that l satisfies the quadratic equation $l = \alpha_l^2 = \frac{1}{4}\big((al/\det h) + d\big)^2$.

On the other hand, if q is even, a given $l \in k^*$ contributes to the second term of (4-6) if and only if $(al + d \det h) = 0$ since $(X - \alpha_l)^2 = X^2 - \alpha_l^2$, and as remarked earlier every l is a square when q is even. $\square$

5. Examples

Take $q = 2$. By (4-1), there is a unique supercuspidal representation π of $\mathrm{GL}_2(\mathbb{Q}_2)$ of conductor 2^2. As mentioned before, the cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$ is the character ρ sending a matrix g to $(-1)^{|g|+1}$, where $|g|$ is the order of g in the finite group. Explicitly, ρ sends $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ to $1, -1, -1, -1, 1, 1$ respectively. This one-dimensional representation of course coincides with its matrix coefficient:

$$\big\langle \rho(h)w, w\big\rangle = \rho(h)\langle w, w\rangle = \rho(h).$$

Indeed, one may verify that Theorem 3.2 recovers ρ when $q = 2$. For example, consider $h = \left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$. The polynomial (3-8) becomes $p_{h,1}(X) = X^2 + X + 1$. If $\theta \in \mathbb{F}_4^*$ is a root, then $\nu(\theta) = \exp(2\pi i/3)$ defines a primitive character. By (3-9),

$$\rho(h) = -\big(\nu(\theta) - \nu(\theta^2)\big) = 1.$$

By (3-6), the matrix coefficient Φ attached to the new vector of π has the simple expression

$$\Phi(g) = \begin{cases} \rho(h) & \text{if } g = z\left(\begin{smallmatrix}2^{-1}&\\&1\end{smallmatrix}\right)h\left(\begin{smallmatrix}2&\\&1\end{smallmatrix}\right) \in Z\left(\begin{smallmatrix}2^{-1}&\\&1\end{smallmatrix}\right)K\left(\begin{smallmatrix}2&\\&1\end{smallmatrix}\right), \\ 0 & \text{otherwise.} \end{cases} \tag{5-1}$$

Now consider $k = \mathbb{F}_5$ and $L = \mathbb{F}_{25}$. Then $L = k[\theta]$, where $\theta^2 = 2$. One finds that $1 + 2\theta$ generates the cyclic group L^*, so the characters of L^* are the maps

$$\nu_n(1 + 2\theta) = \zeta^n \quad (n \in \mathbb{Z}/24\mathbb{Z}),$$

where $\zeta = \exp(2\pi i/24)$. Note that ν_n is primitive if and only if $6 \nmid n$. There are exactly four characters of k^*, given by

$$\omega_n(3) = i^n \quad (n \in \mathbb{Z}/4\mathbb{Z}).$$

Noting that $\nu_n(3) = \nu_n((1+2\theta)^6) = \zeta^{6n} = i^n$, we see that $\nu_n|_{k^*} = \omega_n$. So the equivalence classes of primitive characters of L^* are as follows:

$$[\omega_0] = \{\nu_4, \nu_{20}, \nu_8, \nu_{16}\}, \quad [\omega_1] = \{\nu_1, \nu_5, \nu_9, \nu_{21}, \nu_{13}, \nu_{17}\},$$
$$[\omega_2] = \{\nu_2, \nu_{10}, \nu_{14}, \nu_{22}\}, \quad [\omega_3] = \{\nu_3, \nu_{15}, \nu_7, \nu_{11}, \nu_{19}, \nu_{23}\},$$

where each primitive character is listed alongside its conjugate. The above illustrates Proposition 2.3. As a simple illustration of Theorem 4.2, we now show that

$$\sum_{\nu \in [\omega_0]} \left\langle \rho_\nu\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) w, w \right\rangle = 0.$$

For $h = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ the polynomial (3-8) becomes $p_{h,l}(X) = X^2 + l$. The second term in (4-6) vanishes. Thus an element $l \in k^*$ contributes to (4-6) only if $-l$ is a quadratic nonresidue in k. The roots of $X^2 + 2$ are

$$\alpha_2 = (1+2\theta)^3 \quad \text{and} \quad \bar{\alpha}_2 = (1+2\theta)^{15}.$$

The roots of $X^2 + 3$ are

$$\alpha_3 = \theta = (1+2\theta)^9 \quad \text{and} \quad \bar{\theta} = (1+2\theta)^{21}.$$

For any $\nu \in [\omega_0]$, $\nu(\alpha_j)$ is a power of $\zeta^{12} = -1$ when $4|n$. Hence $\nu^5(\alpha_j) = \nu(\alpha_j)$. Thus in this case (4-6) equals

$$-2\big[2\nu_4(\alpha_2) + 2\nu_8(\alpha_2) + 2\nu_4(\alpha_3) + 2\nu_8(\alpha_3)\big] = -4[-1+1-1+1] = 0.$$

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knightly@math.umaine.edu *Department of Mathematics and Statistics, University of Maine, 5752 Neville Hall, Room 333, Orono, ME 04469-5752, United States*

cwragsda@syr.edu *Department of Mathematics and Statistics, University of Maine, 5752 Neville Hall, Room 333, Orono, ME 04469-5752, United States*