SIMPLE SUPERCUSPIDAL REPRESENTATIONS OF $\mathrm{GL}(n)$

Andrew Knightly and Charles Li

Abstract. Following a construction of Gross and Reeder, we define simple supercuspidal representations of $\mathrm{GL}(n)$ over a $\mathfrak{p}$-adic field. We show that they have conductor $\mathfrak{p}^{n+1}$. We then give a general formula for the matrix coefficient attached to a new vector, and make it completely explicit when $n = 2$.

1. INTRODUCTION

Let F be a nonarchimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p} = \varpi\mathfrak{o}$, and finite residue field $\Bbbk = \mathfrak{o}/\mathfrak{p}$ of cardinality q. Typically, the first textbook example of a supercuspidal representation is one induced from the lift of a cuspidal representation of $\mathrm{GL}_n(\Bbbk)$. These are the supercuspidals of minimal conductor exponent, namely n. But even when $n = 2$ it is not entirely trivial to explicitly define the cuspidal series. There is an easier construction of supercuspidals singled out recently by Gross and Reeder, who defined *simple supercuspidal representations* for a large class of groups over F ([GR], §9). These are induced from affine generic characters χ of a pro-unipotent radical of a maximal compact subgroup. The construction of [GR] applies in particular to $\mathrm{SL}_n(F)$, and our goal here is to treat the case of $\mathrm{GL}_n(F)$. In this case, the representation induced from χ is reducible, decomposing into n irreducible summands (see Theorem 4.4). These summands may naturally be termed simple supercuspidal representations of $\mathrm{GL}_n(F)$. We describe them explicitly and give some of their properties. In particular, there are exactly $n(q-1)$ of them with a given central character, up to isomorphism. These representations are of interest in large part because of their ease of access, being induced from characters (cf. (4.9)).

The construction of all supercuspidal representations of $\mathrm{GL}_n(F)$ has been known since the work of Bushnell and Kutzko in the late 1980's, [BK]. The earliest systematic treatment seems to be that of Carayol in the late 1970's, [C1, C2]. In particular, the

representations constructed here are of the general type described in the third section of [C1]. So in this sense, much of what we discuss in this paper is "well-known". Nevertheless, it seems worthwhile to give a self-contained account, particularly since simple supercuspidal representations are appearing more frequently in the literature as an accessible class of examples. For instance, Adrian and Liu have given a simple proof of the local Langlands correspondence for these representations, [AL].

In our first main result, Corollary 5.2, we show that the conductor of any simple supercuspidal representation of $\mathrm{GL}_n(F)$ is $\mathfrak{p}^{n+1}$, and we exhibit the new vector explicitly. In §6, we then give a general formula for the associated matrix coefficient, which we make completely explicit for the case $n = 2$ in Theorem 7.1. We also show in Proposition 7.2 that any irreducible admissible representation of $\mathrm{GL}_2(F)$ of conductor $\mathfrak{p}^3$ whose central character has at most tame ramification is a simple supercuspidal representation. In §8, we prove Corollary 5.2, and in addition we give an explicit description of the oldforms.

Our motivation for §6 comes from the possibility of using such a matrix coefficient in the (global) trace formula. This gives rise to an operator having purely cuspidal image and isolating those cusp forms with simple supercuspidal $\mathfrak{p}$-type. Gross used this method (but for very general G) to compute multiplicities of cuspidal automorphic representations with certain prescribed local behavior [G]. The local test vector used by Gross is not a new vector: it depends only on χ, and it simultaneously detects the n associated simple supercuspidal representations. However, if one wishes to go further and access the Fourier coefficients or other spectral data, it is necessary to use a new vector. In [KL2], we apply our results to study the L-functions of various newforms of cubic level.

The new vector matrix coefficient of Theorem 7.1 is essentially a Kloosterman sum determined by the matrix entries of the argument. There is a deeper connection between simple supercuspidal representations and Kloosterman sums in the function field setting, described by Heinloth, Ngô and Yun, [HNY]. Their work was motivated by the paper [GF] of Gross and Frenkel.

After this paper was written, the preprint [BH2] of Bushnell and Henniart appeared, in which the Langlands parameters of supercuspidal representations of conductor $\mathfrak{p}^{n+1}$ are described explicitly. We note that these are precisely the representations considered here (cf. Corollary 5.3).

2. AFFINE GENERIC CHARACTERS

Let $G = \mathrm{GL}_n(F)$, let Z be the center of G, identified with F^*, and let $K = \mathrm{GL}_n(\mathfrak{o})$ be the standard maximal compact subgroup. Let M denote the diagonal subgroup of G.

Let $I_n \in G$ be the identity matrix, and let

$$K' = I_n + \begin{pmatrix} \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \cdots & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{o} & \cdots & \mathfrak{o} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & & \cdots & \mathfrak{p} \end{pmatrix}.$$

Relative to the surjective homomorphism $K \longrightarrow \mathrm{GL}_n(\Bbbk)$, K' is the preimage of the upper triangular unipotent subgroup $N(\Bbbk)$, which is a p-Sylow subgroup of $\mathrm{GL}_n(\Bbbk)$ for $p = \mathrm{char}(\Bbbk)$. We set

$$H = ZK'.$$

We will define a simple class of characters of H. Fix a character ω of Z, trivial on $1 + \mathfrak{p}$. Let

$$\psi : \Bbbk \longrightarrow \mathbf{C}^*$$

be a fixed nontrivial additive character. Then every character of $\Bbbk$ is of the form $x \mapsto \psi(tx)$ for some $t \in \Bbbk$. We freely identify these characters with their pullbacks to functions on $\mathfrak{o}$. For $t_1, \ldots, t_n \in \Bbbk^*$, define a function $\chi : H \longrightarrow \mathbf{C}^*$ by

(2.1) $$\chi(zk) = \omega(z)\psi(t_1 r_1 + \cdots + t_n r_n),$$

for $z \in Z$ and

(2.2) $$k = \begin{pmatrix} x_1 & r_1 & * & \cdots & \\ * & x_2 & r_2 & \cdots & \\ \vdots & & \ddots & \ddots & \\ * & & & & r_{n-1} \\ \varpi r_n & & \cdots & & x_n \end{pmatrix} \in K'.$$

It is easy to check that χ is a continuous homomorphism. It is well-defined since ω is trivial on

$$Z \cap K' \cong 1 + \mathfrak{p}.$$

These are the **affine generic characters** of H ([C1],[GR]). For example, if $n = 2$, $F = \mathbf{Q}_p$, and $\omega = 1$, then they are of the form

$$\chi(zk) = e^{\frac{2\pi i}{p}(t_1 b + t_2 c)} \qquad (k = \begin{pmatrix} a & b \\ pc & d \end{pmatrix} \in K', z \in Z).$$

Generally, there are $(q-1)^n$ affine generic characters with a given central character ω.

For any $i \neq j$ (taken modulo n), let $U_{i,j} \subset K'$ denote the subgroup of unipotent matrices with zeroes off the diagonal except in the (i,j)-th slot. Affine generic characters are required by definition to be nontrivial on the n subgroups $U_{i,i+1}$, and trivial on all other $U_{i,j}$.

In the language of [BK], the character χ of K' corresponds to the simple stratum

(2.3) $$[\mathfrak{A}, 1, 0, \beta],$$

where $\mathfrak{A} = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \cdots & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} & \mathfrak{o} & \cdots & \mathfrak{o} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & & \cdots & \mathfrak{o} \end{pmatrix}$ is the standard minimal order of $M_n(F)$, and in the notation

of Lemma 3.2 below, $\beta = t_n g_\chi^{-1} = \begin{pmatrix} 0 & & & & \frac{t_n}{\varpi} \\ t_1 & & & & \\ & t_2 & & & \\ & & \ddots & & \\ & & & t_{n-1} & 0 \end{pmatrix}$. Indeed, this corresponds to

the character of the group $U^1(\mathfrak{A}) = K'$ given by

$$\psi_\beta(x) = \psi(\operatorname{tr}(\beta(x-1))) = \chi(x).$$

3. The Induced Representation

For an affine generic character χ, define the compactly induced representation:

$$\pi_\chi = \text{c-Ind}_H^G(\chi).$$

This is the action of G by right translation on the space A of functions $f : G \longrightarrow \mathbf{C}$ satisfying:

(a) $f(hg) = \chi(h)f(g)$ for all $h \in H$ and $g \in G$

(b) The support of f is compact modulo H, or equivalently, modulo Z

(c) f is smooth, i.e. there exists a compact open subgroup J_f of G such that f is constant on all cosets gJ_f.

Proposition 3.1. *Any irreducible subrepresentation of π_χ is supercuspidal.*

Proof. Let (π, W) be an irreducible subrepresentation of π_χ, and let $\phi \in W$ be a nonzero vector. There exists an open compact subgroup $J \subset G$ such that ϕ is J-invariant. Fixing a Haar measure dx on $\overline{G} = G/Z$, it follows that the linear functional $\check{\phi} : W \to \mathbf{C}$ defined by

$$\check{\phi} : \eta \mapsto \langle \eta, \phi \rangle := \int_{\overline{G}} |\omega(\det x)|^{-2/n} \eta(x)\overline{\phi(x)}dx$$

is J-invariant for the action $\pi^*(g)\check{\phi} = (\pi(g)\phi)\check{}$. Hence it belongs to the smooth dual $\check{W} = \bigcup_J (W^*)^J$. The matrix coefficient

$$g \mapsto \check{\phi}(\pi(g)\eta) = \langle \pi(g)\eta, \phi \rangle = \int_{\overline{G}} |\omega(\det x)|^{-2/n} \eta(xg)\overline{\phi(x)}dx$$

is supported in $\operatorname{Supp}(\phi)^{-1}\operatorname{Supp}(\eta)$, which is compact modulo Z. Since π is smooth, this shows that it is supercuspidal. ∎

The diagonal group $M \cap K$ normalizes H. Therefore it acts on the set of affine generic characters by the rule

$$\chi^m(x) = \chi(mxm^{-1}).$$

Explicitly, if $\chi \leftrightarrow \langle t_1, \ldots, t_n \rangle$ as in (2.1), and $m = \mathrm{diag}(y_1, \ldots, y_n)$, then identifying the y_i with their images in $\Bbbk^*$, we have

$$(3.1) \qquad \chi^m \leftrightarrow \left\langle t_1 \tfrac{y_1}{y_2}, t_2 \tfrac{y_2}{y_3}, \ldots, t_n \tfrac{y_n}{y_1} \right\rangle.$$

By taking $y_1 = 1$, $y_2 = t_1 y_1$, $y_3 = t_2 y_2$, etc., we see that the orbit of χ has a unique element of the form $\langle 1, \ldots, 1, t \rangle$. In fact, $t = t_1 t_2 \cdots t_n$. Thus there are exactly $(q-1)$ orbits of affine generic characters with a given central character.

We will show that given two affine generic characters χ and η, the representations they induce are isomorphic if and only if χ and η belong to the same orbit. See Corollary 4.3 below.

Lemma 3.2. *For $t_1, \ldots, t_n \in \mathfrak{o}^*$, let $\chi \leftrightarrow \langle t_1, \ldots, t_n \rangle$ be the associated affine generic character of H. Let $m = \mathrm{diag}(t_n/t_1, \ldots, t_n/t_{n-1}, 1) \in G$, and define*

$$g_\chi = m \begin{pmatrix} 0 & I_{n-1} \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tfrac{t_n}{t_1} & & & \\ & & \tfrac{t_n}{t_2} & & \\ & & & \ddots & \\ & & & & \tfrac{t_n}{t_{n-1}} \\ \varpi & & & & 0 \end{pmatrix} \in G.$$

Then g_χ normalizes K' (thus also H) and $\chi = \chi^{g_\chi}$, i.e.

$$\chi(g_\chi^{-1} h g_\chi) = \chi(h)$$

for all $h \in H$.

Remarks.

(a) g_χ^n is the scalar matrix $\frac{t_n^{n-1}\varpi}{t_1 t_2 \cdots t_{n-1}} I_n$.

(b) We showed above that every orbit of characters contains one of the form $\chi \leftrightarrow \langle 1, \ldots, 1, t \rangle$. In this case g_χ is simply

$$(3.2) \qquad g_\chi = \begin{pmatrix} tI_{n-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 0 & I_{n-1} \\ \varpi & 0 \end{pmatrix} = \begin{pmatrix} 0 & tI_{n-1} \\ \varpi & 0 \end{pmatrix}.$$

(c) The matrix g_χ is not uniquely determined by χ because it involves choosing representatives for $t_1, \ldots, t_n \in \mathfrak{o}^*/(1+\mathfrak{p})$. This choice is immaterial and we regard it as fixed once and for all.

Proof. It suffices to consider $h \in K'$. Writing $w_\varpi = \left(\begin{smallmatrix} & I_{n-1} \\ \varpi & \end{smallmatrix} \right)$, we have

$$
w_\varpi^{-1}
\begin{pmatrix}
\frac{t_1}{t_n} & & & & \\
& \frac{t_2}{t_n} & & & \\
& & \ddots & & \\
& & & \frac{t_{n-1}}{t_n} & \\
& & & & 1
\end{pmatrix}
\begin{pmatrix}
x_1 & r_1 & * & \cdots \\
* & x_2 & r_2 & \cdots \\
\vdots & & \ddots & \ddots \\
* & & & r_{n-1} \\
\varpi r_n & \cdots & & x_n
\end{pmatrix}
\begin{pmatrix}
\frac{t_n}{t_1} & & & & \\
& \frac{t_n}{t_2} & & & \\
& & \ddots & & \\
& & & \frac{t_n}{t_{n-1}} & \\
& & & & 1
\end{pmatrix}
w_\varpi =
$$

$$
\left(\begin{smallmatrix} I_{n-1} & \\ & \frac{1}{\varpi} \end{smallmatrix} \right)
\begin{pmatrix}
x_1 & \frac{t_1 r_1}{t_2} & * & \cdots \\
* & x_2 & \frac{t_2 r_2}{t_3} & \cdots \\
\vdots & & \ddots & \ddots \\
* & & & \frac{t_{n-1} r_{n-1}}{t_n} \\
\varpi \frac{t_n r_n}{t_1} & \cdots & & x_n
\end{pmatrix}
\left(\begin{smallmatrix} & I_{n-1} \\ \varpi & \end{smallmatrix} \right)
=
\begin{pmatrix}
x_n & \frac{t_n r_n}{t_1} & * & \cdots \\
* & x_1 & \frac{t_1 r_1}{t_2} & \cdots \\
\vdots & & \ddots & \ddots \\
* & & & \\
\varpi \frac{t_{n-1} r_{n-1}}{t_n} & \cdots & & x_{n-1}
\end{pmatrix},
$$

where all entries in the latter matrix are integral and those below the diagonal belong to $\mathfrak{p}$. Thus

$$
\chi(g_\chi^{-1} h g_\chi) = \psi(t_n r_n + t_1 r_1 + \cdots + t_{n-1} r_{n-1}) = \chi(h). \qquad \blacksquare
$$

In order to understand (π_χ, A), it is useful to determine the subspaces

$$
A^\eta = \{ v \in A \,|\, \pi_\chi(h)v = \eta(h)v \text{ for all } h \in H \},
$$

for various affine generic characters η. There is an obvious nonzero element of A^χ, namely the function

$$
(3.3) \qquad f_0(h) = \begin{cases} \chi(h) & \text{if } h \in H \\ 0 & \text{otherwise.} \end{cases}
$$

We will show below that this function and its left translates by g_χ^{-1} span A^χ.

Proposition 3.3. *Let χ and g_χ be as above, and let η be any affine generic character of H. Suppose $\phi \in A^\eta$. If $\phi(x) \neq 0$, then*

$$
(3.4) \qquad \eta(h) = \chi(xhx^{-1}) \quad \text{for all} \quad h \in H \cap x^{-1}Hx.
$$

This condition is independent of the choice of representative x for the double coset HxH. Conversely, if $x \in G$ is any element satisfying (3.4), then there exists a unique element $\phi_x \in A^\eta$ supported on HxH and satisfying $\phi_x(x) = 1$.

An element x satisfies (3.4) if and only if $g_\chi x$ satisfies (3.4). For such x, the set

$$
(3.5) \qquad \{ \phi_x, \phi_{g_\chi x}, \ldots, \phi_{g_\chi^{n-1} x} \} \subset A^\eta
$$

is linearly independent.

Proof. Suppose $\phi(x) \neq 0$ and $h \in H \cap x^{-1}Hx$. Then

$$\eta(h)\phi(x) = \phi(xh) = \phi(xhx^{-1}x) = \chi(xhx^{-1})\phi(x),$$

so (3.4) is satisfied. Conversely, suppose x satisfies (3.4). Define

$$\phi_x(g) = \begin{cases} \chi(h_1)\eta(h_2) & \text{if } g = h_1xh_2 \text{ for some } h_1, h_2 \in H \\ 0 & \text{otherwise.} \end{cases}$$

Clearly this belongs to A^η provided it is well-defined, i.e. that

(3.6) $$\chi(h_1)\eta(h_2) = \chi(h'_1)\eta(h'_2)$$

whenever $h_1xh_2 = h'_1xh'_2$ for $h_i, h'_i \in H$. In this case,

$$x^{-1}{h'_1}^{-1}h_1x = h'_2h_2^{-1} \in H \cap x^{-1}Hx,$$

so by (3.4), $\chi({h'_1}^{-1}h_1) = \eta(h'_2h_2^{-1})$, which immediately gives (3.6).

Next, note that because $g_\chi^{-1}Hg_\chi = H$, $H \cap x^{-1}Hx = H \cap (g_\chi x)^{-1}Hg_\chi x$. Furthermore, for all h in this set, we have

$$\chi(xhx^{-1}) = \chi(g_\chi xhx^{-1}g_\chi^{-1})$$

by Lemma 3.2. It follows that x satisfies (3.4) if and only if $g_\chi x$ does.

Lastly, the determinants of $x, g_\chi x, \ldots, g_\chi^{n-1}x$ have valuations which are distinct mod n, by which we see that the functions in (3.5) have pairwise disjoint supports. Thus they are linearly independent. ∎

Theorem 3.4. *Suppose $\chi \leftrightarrow \langle t_1, \ldots, t_n \rangle$ and $\eta \leftrightarrow \langle \ell_1, \ldots, \ell_n \rangle$ as in (2.1). Then the following are equivalent.*

(1) $A^\eta \neq 0$.

(2) $t_1 t_2 \cdots t_n = \ell_1 \ell_2 \cdots \ell_n$ *in* $\Bbbk^*$.

(3) χ *and* η *are in the same orbit, i.e.* $\chi^m = \eta$ *for some* $m \in M \cap K$.

If these conditions hold, then $\dim A^\eta = n$. *In fact, let* $x = \mathrm{diag}(y_1, y_2, y_3, \ldots, y_n)$, *where* $y_1 = 1$ *and* $y_{i+1} = \frac{y_i t_i}{\ell_i}$. *Then* $\chi^x = \eta$ *and the set* (3.5) *is a basis for* A^η.

Remark. The implication (1) $\implies$ (3) is a special case of Theorem (2.6.1) of [BK]. Indeed, the condition $A^\eta \neq 0$ means that the strata attached to χ and η intertwine, and so by *loc. cit.*, η and χ are conjugate under $\mathfrak{A}^* = (M \cap K)K'$. For convenience, we include an elementary proof below.

Proof. First we prove that (1) implies (3). Suppose $\phi \in A^\eta$ is nonzero. We can assume that the support of ϕ is a double coset HxH, where x satisfies (3.4). We can also take $\phi(x) = 1$. Let T be the normalizer in G of the diagonal subgroup M. Then

$$T = \bigcup_{w \in \mathcal{W}} Mw = \bigcup_{w \in \mathcal{W}} wM,$$

where $\mathcal{W}$ is the Weyl group of G consisting of those matrices with exactly one 1 in each row and column, and 0's everywhere else. Recall the affine Bruhat decomposition

$$G = K'TK'$$

([Ho], p. 77). In particular, by Proposition 3.3 we can assume that

$$x = wm$$

for some $w \in \mathcal{W}$ and $m \in M$.

By Proposition 3.3, we are also free to replace x by $g_\chi^a x$ for any $a \geq 0$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for F^n. The i-th row of w is $e_{\sigma(i)}$ for a permutation $\sigma \in S_n$. The Weyl element $\left(\begin{smallmatrix} & I_{n-1} \\ 1 & \end{smallmatrix}\right)$ used in the definition of g_χ corresponds in this way to the n-cycle $(1\ 2\ 3 \ldots n) \in S_n$. Hence we can choose a in such a way that the resulting element $x = wm$ has the property that the permutation σ associated to w fixes the number n.

With this choice of x, we claim that $w = 1$. Write $m = \mathrm{diag}(y_1, \ldots, y_n)$. Let $b_j = \mathrm{ord}_{\mathfrak{p}}(y_j)$. Suppose $w \neq 1$, so $\sigma \neq 1$. We will derive a contradiction. Let $1 < \ell \leq n$ be the smallest integer with the property that σ fixes $\ell, \ell + 1, \ldots, n$. Thus $\sigma(\ell - 1) < \ell - 1$. Adjusting x by an element of the center if necessary, we can assume that $y_\ell = 1$, so $b_\ell = 0$. We prove the claim in the following steps:

(i) Show that $b_{\sigma(\ell-1)} > 0$.

(ii) Show that $b_{\sigma(j+1)} \leq b_{\sigma(j)}$ for all j, with indices taken modulo n.

Finally, step (ii) implies that

$$b_{\sigma(1)} \geq b_{\sigma(2)} \geq \cdots \geq b_{\sigma(n)} \geq b_{\sigma(1)},$$

so that all b_j are equal. In particular $b_\ell = b_{\sigma(\ell-1)} > 0$, contradicting $b_\ell = 0$. Let E_{ij} be the $n \times n$ matrix whose only non-zero entry is a 1 in the i-th row and j-th column. We regard i, j as indices modulo n. Note that

$$x^{-1}E_{ij}x = m^{-1}w^{-1}E_{ij}wm = m^{-1}E_{\sigma(i)\sigma(j)}m = \frac{y_{\sigma(j)}}{y_{\sigma(i)}}E_{\sigma(i)\sigma(j)}.$$

To prove (i), let

$$k = I_n + E_{\ell-1,\ell} \in K'.$$

If $b_{\sigma(\ell-1)} \leq 0$, then

$$h = x^{-1}kx = I_n + \tfrac{1}{y_{\sigma(\ell-1)}}E_{\sigma(\ell-1),\ell} \in K'.$$

The nonzero entry of $E_{\sigma(\ell-1),\ell}$ is at least two positions above the main diagonal, so $\eta(h) = 1$. By (3.4), we have $1 = \eta(h) = \chi(k) = \psi(t_{\ell-1})$, a contradiction. This proves (i). To prove (ii), fix any $j \neq n$ and define

$$k = I_n + E_{j,j+1} \in K'.$$

If $b_{\sigma(j+1)} > b_{\sigma(j)}$, then $y_{\sigma(j+1)}/y_{\sigma(j)} \in \mathfrak{p}$, so

$$h = x^{-1}kx = I_n + \tfrac{y_{\sigma(j+1)}}{y_{\sigma(j)}} E_{\sigma(j)\sigma(j+1)} \in K'.$$

Therefore by (3.4), $\eta(h) = \chi(k)$. Since $j \neq n$, we have $\eta(h) = 1$, giving the contradiction $1 = \psi(t_j)$. Now suppose $j = n$, and define $k = I_n + \varpi E_{n,1} \in K'$. If $b_{\sigma(1)} > b_{\sigma(n)}(= b_n)$, then $y_{\sigma(1)}/y_n \in \mathfrak{p}$,

$$h = x^{-1}kx = I_n + \varpi \tfrac{y_{\sigma(1)}}{y_n} E_{n,\sigma(1)} \in K',$$

and $1 = \eta(h) = \chi(k) = \psi(t_n)$, a contradiction. This proves (ii), and therefore we can assume that $x = m$ is diagonal.

Write $x = \mathrm{diag}(y_1, y_2, \ldots, y_n)$. Adjusting x by the center, we can assume that $y_1 = 1$. We will show inductively that each y_j is a unit. Suppose $y_1, \ldots, y_k$ are units, with $k < n$. Consider $h = I_n + rE_{k,k+1}$ for $r \in \mathfrak{o}$. Then

$$xhx^{-1} = I_n + \tfrac{y_k}{y_{k+1}} rE_{k,k+1}.$$

Because x satisfies (3.4), we have

$$\psi(t_k r \tfrac{y_k}{y_{k+1}}) = \psi(\ell_k r)$$

whenever $r \in \mathfrak{o} \cap y_{k+1}\mathfrak{o}$. If $y_{k+1} \in \mathfrak{p}$, then taking $r = y_{k+1}$ gives $\psi(t_k y_k) = \psi(\ell_k y_{k+1}) = 1$. This contradicts the fact that $t_k, y_k \in \mathfrak{o}^*$. If $y_{k+1} \notin \mathfrak{o}$, then taking $r = 1$ gives $\psi(\ell_k) = \psi(t_k y_k y_{k+1}^{-1}) = 1$, another contradiction. The only possibility remaining is that $y_{k+1} \in \mathfrak{o}^*$ and

$$(3.7) \qquad\qquad y_{k+1} \equiv \tfrac{y_k t_k}{\ell_k} \bmod \mathfrak{p}.$$

In particular, by induction $x \in M \cap K$. Therefore $H \cap x^{-1}Hx = H$, so (3.4) says exactly that (3) holds with $m = x$.

The implication (3) $\implies$ (1) is immediate from Proposition 3.3.

The equivalence of (2) and (3) is immediate from the discussion following (3.1).

Note that in the proof of (1) $\implies$ (3), the entries of the diagonal matrix x are uniquely determined modulo $(1+\mathfrak{p})$ by (3.7). In view of the last assertion of Proposition 3.3, it follows that (3.5) is a basis for A^η. ∎

4. Decomposition of π_χ

As before, let χ be an affine generic character of H. In Theorem 4.4 below, we will prove that the induced representation π_χ is the direct sum of n distinct supercuspidal representations. These are parametrized naturally by the pairs (t, ζ), where t is a nonzero element of the residue field (determining the orbit of χ), and ζ is a complex n^{th} root of $\omega(t^{n-1}\varpi)$ (identifying one of the irreducible summands of π_χ).

4.1. Preliminaries

Let
$$E = E(F) = \{g \in G|\ \mathrm{ord}_\mathfrak{p}(\det g) \in n\mathbf{Z}\}.$$
This is a normal subgroup of G of index n, containing both H and $\mathrm{SL}_n(F)$. Note that G is the disjoint union

(4.1)
$$G = E \cup Eg_\chi \cup \cdots \cup Eg_\chi^{n-1}.$$

Accordingly, there is a decomposition
$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{n-1},$$

where A_k consists of functions supported on Eg_χ^k. Indeed, A is spanned by functions of the form
$$\phi(g) = \begin{cases} \chi(h) & \text{if } g = hxk \in HxJ \\ 0 & \text{if } g \notin HxJ \end{cases}$$

for $x \in G$ and J an open compact subgroup. By the fact that $\det J \subset \mathfrak{o}^*$, such a function belongs to A_k if and only if $x \in Eg_\chi^k$. Note that $A_0, \ldots, A_{n-1}$ are closed E-submodules of A. The representation of E on A_0 is precisely the compactly induced representation
$$\sigma_\chi \overset{\text{def}}{=} \text{c-Ind}_H^E(\chi).$$

Proposition 4.1. *The representation (σ_χ, A_0) of E is irreducible. Two such representations σ_χ and σ_η are equivalent if and only if $\eta = \chi^m$ for some $m \in M \cap K$.*

Proof. Let W be any nonzero E-invariant subspace of A_0. By Frobenius reciprocity ([BH1], p. 20),
$$0 \neq \text{Hom}_E(W, \text{c-Ind}_H^E(\chi)) \cong \text{Hom}_H(W, \chi).$$

Therefore W^χ is a nonzero subspace of A_0^χ. The basis for A^χ given by Theorem 3.4 has exactly one element supported on E, namely the function f_0 of (3.3), so $\dim A_0^\chi = 1$ and hence $f_0 \in W$. But it follows immediately from the definition of $\text{c-Ind}_H^E(\chi)$ that f_0 generates A_0 as a $\mathbf{C}[E]$-module, i.e. $A_0 = \mathbf{C}[E] \cdot f_0 \subset W$. Hence $W = A_0$, so σ_χ is irreducible.

If $\sigma_\chi \cong \sigma_\eta$, then there exists a nonzero function $\phi \in A_0^\eta$. By Theorem 3.4, $\eta = \chi^m$ for some $m \in M \cap K$. Conversely, if $\eta = \chi^m$, then by Theorem 3.4

$$\mathrm{Hom}_H(\eta, \sigma_\chi) \cong A_0^\eta \neq 0,$$

so by Frobenius reciprocity, $\mathrm{Hom}_E(\sigma_\eta, \sigma_\chi) \neq 0$. Since σ_η and σ_χ are both irreducible, this implies $\sigma_\eta \cong \sigma_\chi$. ∎

Proposition 4.2. *The representations $A_0, \ldots, A_{n-1}$ of E are each irreducible and isomorphic to the representation $\sigma_\chi = \mathrm{c\text{-}Ind}_H^E(\chi)$.*

Proof. Define an operator on A by

(4.2) $$[L\phi](x) = \phi(g_\chi^{-1} x).$$

The fact that $L\phi \in A$ is a consequence of Lemma 3.2:

$$L\phi(hx) = \phi(g_\chi^{-1} h g_\chi g_\chi^{-1} x) = \chi(g_\chi^{-1} h g_\chi)\phi(g_\chi^{-1} x) = \chi(h)L\phi(x).$$

Furthermore, it is clear that for $k = 0, \ldots, n-1$ the map

$$L : A_k \longrightarrow A_{k+1}$$

is a vector space isomorphism (with subscripts taken modulo n). In fact, because E acts on both spaces by right translation, this isomorphism is E-equivariant. ∎

Corollary 4.3. *Given two affine generic characters χ and η of H, the induced representations π_χ and π_η of G are equivalent if and only if χ and η belong to the same $M \cap K$-orbit.*

Proof. By Proposition 4.1, χ and η belong to the same orbit if and only if $\sigma_\chi \cong \sigma_\eta$. If this holds, then clearly $\pi_\chi \cong \pi_\eta$, since by the transitivity of compact induction,

$$\pi_\chi = \mathrm{c\text{-}Ind}_H^G(\chi) \cong \mathrm{c\text{-}Ind}_E^G(\sigma_\chi).$$

Conversely, if $\pi_\chi \cong \pi_\eta$, then they are also isomorphic as representations of E. By Proposition 4.2, this only possible if $\sigma_\chi \cong \sigma_\eta$. ∎

The corollary shows that we are free to assume that $\chi \leftrightarrow \langle 1, \ldots, 1, t \rangle$ as in (3.1). This is convenient for calculations since g_χ then has the simple form given in (3.2).

4.2. Decomposition of π_χ

We now show that (π_χ, A) is the direct sum of n supercuspidal representations. Here we assume without loss of generality that $\chi \leftrightarrow \langle 1, \ldots, 1, t \rangle$ for $t \in \Bbbk^*$. Let $\zeta \in \mathbf{C}$ satisfy $\zeta^n = \omega(t^{n-1}\varpi)$. Define the subspace

(4.3) $$\Sigma_\zeta = \left\{ \sum_{j=0}^{n-1} (\zeta L)^j \phi \;\middle|\; \phi \in A_0 \right\} \subset A,$$

for the operator L given in (4.2). The map $\phi \mapsto \sum (\zeta L)^j \phi$ from $A_0 \to \Sigma_\zeta$ is an isomorphism of E-modules. Its inverse is given by restriction to E. Thus Σ_ζ is an irreducible representation of E isomorphic to σ_χ. Let $\phi \in A_0$, and set $\psi = \pi_\chi(g_\chi)\phi \in A_{n-1}$. Then $L\psi \in A_0$ and

$$(4.4) \qquad \pi_\chi(g_\chi)[\phi + \zeta L\phi + \cdots + (\zeta L)^{n-1}\phi] = \psi + \zeta L\psi + \cdots + (\zeta L)^{n-1}\psi.$$

Notice that because $g_\chi^n = t^{n-1}\varpi I_n$ is a scalar matrix, for any $\xi \in A$ we have $L^n\xi = \omega(t^{n-1}\varpi)^{-1}\xi$, i.e.

$$(4.5) \qquad \xi = \omega(t^{n-1}\varpi)L^n\xi = (\zeta L)^n\xi.$$

Hence, taking $\xi = \psi$, (4.4) becomes

$$(4.6) \qquad \pi_\chi(g_\chi)\sum_{j=0}^{n-1}(\zeta L)^j\phi = (\zeta L\psi) + \zeta L(\zeta L\psi) + \cdots + (\zeta L)^{n-1}(\zeta L\psi) \in \Sigma_\zeta.$$

Thus Σ_ζ is a G-submodule of A, which is necessarily irreducible since it is irreducible as an E-module. By Proposition 3.1 it is supercuspidal. We denote the action of G on Σ_ζ by σ_χ^ζ. Following Gross and Reeder, we call σ_χ^ζ a **simple supercuspidal representation** of $\mathrm{GL}_n(F)$.

It is easy to check that the sum of the subspaces Σ_ζ (for ζ ranging over the complex n^{th} roots of $\omega(t^{n-1}\varpi)$) is direct. This proves the first part of the following.

Theorem 4.4. *We have*

$$\pi_\chi \cong \bigoplus_{\zeta^n = \omega(t^{n-1}\varpi)} \sigma_\chi^\zeta$$

as representations of G. The representation π_χ is multiplicity-free, i.e. the representations σ_χ^ζ are mutually inequivalent.

Proof. We just need to prove the final assertion. If two or more of the representations σ_χ^ζ were isomorphic, then the dimension of $\mathrm{Hom}_G(\pi_\chi, \pi_\chi)$ would exceed n. However, by Frobenius reciprocity,

$$A^\chi \cong \mathrm{Hom}_H(\chi, \pi_\chi) \cong \mathrm{Hom}_G(\pi_\chi, \pi_\chi),$$

and by Theorem 3.4, $\dim A^\chi = n$. $\blacksquare$

By the isomorphism of A_0 with Σ_ζ, the representation σ_χ^ζ has a model on the space A_0. It is given by

$$(4.7) \qquad \sigma_\chi^\zeta(g)\phi(x) = \zeta^k\phi(g_\chi^{-k}xg) \qquad (\phi \in A_0),$$

where $k \in \{0, \ldots, n-1\}$ is determined by $g \in E g_\chi^k$. The notation is simpler in this model, so we will use it especially in Section 6 when we describe the matrix coefficients of σ_χ^ζ.

4.3. Related considerations

As proven by Bushnell and Kutzko, every supercuspidal representation of G is compactly induced from a finite dimensional representation of a subgroup that is open and compact modulo the center, [BK].

The inducing data for a simple supercuspidal representation is given as follows. The relevant group is $H' = \langle g_\chi \rangle H$. We have

$$(4.8) \qquad \text{c-Ind}_H^{H'}(\chi) = \bigoplus_{\zeta^n = \omega(t^{n-1}\varpi)} \chi_\zeta,$$

where χ_ζ is the character of H' given by $\chi_\zeta(g_\chi^j h) = \zeta^j \chi(h)$. Explicitly, the character χ_ζ is generated by the element $\sum_{j=0}^{n-1} (\zeta L)^j \phi$, where $\phi \in \text{c-Ind}_H^{H'}(\chi)$ is the unique element supported on H with $\phi(I_n) = 1$. Then by transitivity of compact induction,

$$(4.9) \qquad \sigma_\chi^\zeta = \text{c-Ind}_{H'}^G(\chi_\zeta).$$

Secondly, we have seen that the representation (σ_χ, A_0) of E admits n distinct extensions to G, namely the σ_χ^ζ. It is natural to ask whether there are any other extensions. The fact that there do not is a consequence of the following.

Proposition 4.5. *Let* (π, W) *be an irreducible smooth representation of G, and for an affine generic character χ, let*

$$W^\chi = \{v \in W \,|\, \pi(h)v = \chi(h)v \text{ for all } h \in H\}.$$

Then $W^\chi \neq 0$ if and only if $\pi \cong \sigma_\chi^\zeta$ for some ζ.

Proof. By Frobenius reciprocity and Theorem 4.4,

$$W^\chi \cong \text{Hom}_H(\chi, \pi) \cong \text{Hom}_G(\pi_\chi, \pi) \cong \bigoplus_\zeta \text{Hom}_G(\sigma_\chi^\zeta, \pi).$$

The proposition now follows by Schur's Lemma. ∎

Now if (π, W) is any smooth representation of G with $\pi|_E \cong \sigma_\chi$, then π is irreducible and W^χ is nonzero. By the proposition, $\pi \cong \sigma_\chi^\zeta$ for some ζ.

5. The Conductor of σ_χ^ζ

Let $K_1(\mathfrak{p}^m)$ denote the subgroup of K consisting of those matrices with bottom row congruent to $(0, \ldots, 0, 1)$ mod $\mathfrak{p}^m$. By a well-known result of Jacquet, Piatetski-Shapiro and Shalika, for any irreducible admissible generic representation π of $\text{GL}_n(F)$,

there exists a unique integral ideal $\mathfrak{p}^m$ (the **conductor** of π) such that $\dim \pi^{K_1(\mathfrak{p}^m)} = 1$ and $\pi^{K_1(\mathfrak{p}^{m-1})} = \{0\}$ ([JPSS],[J]). A nonzero $K_1(\mathfrak{p}^m)$-fixed vector in the space of π is called a **new vector**. We note that a supercuspidal representation is generic ([GK], Theorem B).

Theorem 5.1. *Let χ be an affine generic character of H, and let $A = \text{c-Ind}_H^G(\chi)$. Among all nonzero $K_1(\mathfrak{p}^{n+1})$-invariant functions in A, there is exactly one (up to multiples) with support of the form $HdK_1(\mathfrak{p}^{n+1})$ for d diagonal. It is supported on*

$$
(5.1) \qquad H \begin{pmatrix} \varpi^{n-1} & & & & \\ & \varpi^{n-2} & & & \\ & & \ddots & & \\ & & & \varpi & \\ & & & & 1 \end{pmatrix} K_1(\mathfrak{p}^{n+1}),
$$

and we denote it by ξ when normalized so that $\xi(\text{diag}(\varpi^{n-1}, \ldots, \varpi, 1)) = 1$. The subspace of $K_1(\mathfrak{p}^{n+1})$-fixed vectors in A is spanned by $\{\xi, L\xi, L^2\xi, \ldots, L^{n-1}\xi\}$. Furthermore, A does not contain a nonzero $K_1(\mathfrak{p}^n)$-invariant function.

We prove Theorem 5.1 in §8 below, where we compute $A^{K_1(\mathfrak{p}^m)}$ for all m.

Corollary 5.2. *The conductor of σ_χ^ζ is equal to $\mathfrak{p}^{n+1}$. For $\chi \leftrightarrow \langle 1, \ldots, 1, t \rangle$, in the model (4.7) for σ_χ^ζ on A_0, the new vector is*

$$
\phi_\mathfrak{p} = \begin{cases} \xi & \text{if } n \text{ is odd} \\ L^{\frac{n}{2}}\xi & \text{if } n \text{ is even,} \end{cases}
$$

where ξ is the function defined in the above theorem and L is defined in (4.2).

Remark. The conductor was computed differently by Carayol ([C2], 4.2 Théorème).

Proof. Note that $H, K_1(\mathfrak{p}^{n+1}) \subset E$. The diagonal matrix in (5.1) has determinant $\varpi^{\frac{n(n-1)}{2}}$, so the assertion follows from the fact that $\frac{n(n-1)}{2} \in n\mathbf{Z}$ if n is odd, while $\frac{n(n-1)}{2} + \frac{n}{2} \in n\mathbf{Z}$ if n is even. $\blacksquare$

For example, when $n = 3$, a basis for $A^{K_1(\mathfrak{p}^4)}$ is given by the functions supported respectively on

$$
H \begin{pmatrix} \varpi^2 & & \\ & \varpi & \\ & & 1 \end{pmatrix} K_1(\mathfrak{p}^4), \quad Hg_\chi \begin{pmatrix} \varpi^2 & & \\ & \varpi & \\ & & 1 \end{pmatrix} K_1(\mathfrak{p}^4), \quad Hg_\chi^2 \begin{pmatrix} \varpi^2 & & \\ & \varpi & \\ & & 1 \end{pmatrix} K_1(\mathfrak{p}^4),
$$

the first one $\phi_\mathfrak{p}$ being the new vector of σ_χ^ζ in its model on A_0.

Corollary 5.3. *Every supercuspidal representation of $\text{GL}_n(F)$ of conductor $\mathfrak{p}^{n+1}$ is isomorphic to a simple supercuspidal representation.*

Proof. Bushnell and Henniart have shown that there are exactly $n(q-1)$ supercuspidal representations of conductor $\mathfrak{p}^{n+1}$ with a given central character, up to isomorphism (see Remark 2.2 of [BH2]). This is also the number of simple supercuspidal representations: $(q-1)$ choices for χ, and n choices for ζ. (The correspondence can be made precise by matching the associated simple strata as in (2.3).) ∎

6. MATRIX COEFFICIENTS

For the next two sections, we suppose that $\chi \leftrightarrow \langle 1, \ldots, 1, t \rangle$ and that ω (and hence π_χ) is unitary. Using the fact that

$$G = E \cup Eg_\chi \cup \cdots \cup Eg_\chi^{n-1},$$

for $\phi, \lambda \in A$, we have the G-invariant inner product

$$(6.1) \quad \langle \pi_\chi(g)\phi, \lambda \rangle = \int_{\overline{G}} \pi_\chi(g)\phi(x)\overline{\lambda(x)}dx = \mathrm{meas}(\overline{H}) \sum_{k=0}^{n-1} \sum_{x \in H \backslash E} \phi(xg_\chi^k g)\overline{\lambda(xg_\chi^k)},$$

where $\overline{G} = G/Z$ and $\overline{H} = H/Z$. This restricts to give a G-invariant inner product on the subrepresentation $(\sigma_\chi^\zeta, \Sigma_\zeta)$. We can then transfer it to the model (σ_χ^ζ, A_0). We will denote by $\langle \phi, \lambda \rangle_0$, $\|\phi\|_0$, etc., the G-invariant inner product on (σ_χ^ζ, A_0) obtained in this way. This inner product on A_0 does *not* coincide with the one A_0 inherits from (6.1) as a subspace of A. For $\phi, \lambda \in A_0$, we have

$$\mathrm{meas}(\overline{H})^{-1} \left\langle \sigma_\chi^\zeta(g)\phi, \lambda \right\rangle_0 = \mathrm{meas}(\overline{H})^{-1} \left\langle \pi_\chi(g) \sum_{\ell=0}^{n-1}(\zeta L)^\ell \phi, \sum_{j=0}^{n-1}(\zeta L)^j \lambda \right\rangle$$

$$= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-1} \sum_{x \in H \backslash E} (\zeta L)^\ell \phi(xg_\chi^k g)\overline{(\zeta L)^j \lambda(xg_\chi^k)}.$$

Define $r \in \{0, \ldots, n-1\}$ by $g \in Eg_\chi^r$. Since ϕ and λ are supported in E, the summand vanishes unless $-\ell + k + r \equiv 0 \bmod n$ and $-j + k \equiv 0 \bmod n$. By (4.5), $(\zeta L)^\ell \phi = (\zeta L)^{k+r} \phi$ when $\ell \equiv k+r \bmod n$. Therefore

$$\mathrm{meas}(\overline{H})^{-1} \left\langle \sigma_\chi^\zeta(g)\phi, \lambda \right\rangle_0 = \sum_{k=0}^{n-1} \sum_{x \in H \backslash E} (\zeta L)^{k+r} \phi(xg_\chi^k g)\overline{(\zeta L)^k \lambda(xg_\chi^k)}$$

$$= \sum_{k=0}^{n-1} \sum_{x \in H \backslash E} |\zeta|^{2k} \zeta^r \phi(g_\chi^{-r} g_\chi^{-k} xg_\chi^k g)\overline{\lambda(g_\chi^{-k} xg_\chi^k)}.$$

Since we assumed ω to be unitary, $|\zeta| = 1$. Using the fact that $g_\chi^k(H\backslash E)g_\chi^{-k} = H\backslash E$, we can replace x by $g_\chi^k x g_\chi^{-k}$ in the inner sum. We obtain:

$$(6.2) \qquad \mathrm{meas}(\overline{H})^{-1} \left\langle \sigma_\chi^\zeta(g)\phi, \lambda \right\rangle_0 = n\zeta^r \sum_{x \in H\backslash E} \phi(g_\chi^{-r}xg)\overline{\lambda(x)} \quad (g \in Eg_\chi^r).$$

For example, let $f_0 \in A_0^\chi$ be the nonzero vector defined in (3.3). Since f_0 is supported on H, taking $g = 1$ and $\phi = \lambda = f_0$ in (6.2) we find

$$(6.3) \qquad \|f_0\|_0^2 = n\,\mathrm{meas}(\overline{H}).$$

Note that this is independent of ζ.

Proposition 6.1. *The formal degree of σ_χ^ζ is independent of ζ, so we denote it by d_χ. It is given by*

$$(6.4) \qquad d_\chi = \frac{1}{n\,\mathrm{meas}(\overline{H})}.$$

Remark. Under the normalization $\mathrm{meas}(\overline{K}) = 1$, we compute $\mathrm{meas}(\overline{H})$ in the proof of Corollary 6.5 below. See also §5 of [C2].

Proof. Let f_0 be as above, and define the matrix coefficient

$$\phi(g) = \left\langle \sigma_\chi^\zeta(g)f_0, f_0 \right\rangle_0.$$

If $g \in g_\chi^r E$ then by (6.2),

$$\phi(g) = n\,\mathrm{meas}(\overline{H})\zeta^r \sum_{x \in H\backslash E} f_0(g_\chi^{-r}xg)\overline{f_0(x)} = n\,\mathrm{meas}(\overline{H})\zeta^r f_0(g_\chi^{-r}g),$$

since as before, only $x = 1$ contributes to the sum. Thus for any g,

$$(6.5) \qquad \phi(g) = \begin{cases} n\,\mathrm{meas}(\overline{H})\zeta^r\chi(h) & \text{if } g = g_\chi^r h \in g_\chi^r H \\ 0 & \text{if } g \notin \bigcup_{r=0}^{n-1} g_\chi^r H. \end{cases}$$

By the orthogonality relations that define d_χ,

$$\int_{\overline{G}} |\phi(g)|^2 dg = \frac{\|f_0\|_0^4}{d_\chi} = \frac{n^2\,\mathrm{meas}(\overline{H})^2}{d_\chi}$$

by (6.3). By (6.5), ϕ is supported on $\bigcup_{r=0}^{n-1} g_\chi^r H$. Thus

$$\int_{\overline{G}} |\phi(g)|^2 dg = \sum_{r=0}^{n-1} \int_{\overline{H}} |\phi(g_\chi^r h)|^2 dh = \sum_{r=0}^{n-1} n^2 \operatorname{meas}(\overline{H})^3 = n^3 \operatorname{meas}(\overline{H})^3.$$

The proposition now follows. ∎

We are particularly interested in the case where $\phi = \lambda = \phi_{\mathfrak{p}}$ is the new vector defined in Corollary 5.2.

Proposition 6.2. *Suppose $g \in E g_\chi^r$ where $0 \le r \le n - 1$. Then*

(6.6) $$d_\chi \left\langle \sigma_\chi^\zeta(g) \phi_{\mathfrak{p}}, \phi_{\mathfrak{p}} \right\rangle_0 = \zeta^r \sum_{x \in H \backslash \operatorname{Supp}(\phi_{\mathfrak{p}})} \phi_{\mathfrak{p}}(g_\chi^{-r} x g) \overline{\phi_{\mathfrak{p}}(x)},$$

Proof. This is immediate from (6.2) and (6.4). ∎

In order to compute (6.6), we need an explicit set of representatives for $H \backslash \operatorname{Supp}(\phi_{\mathfrak{p}})$. In particular this will allow us to compute $\|\phi_{\mathfrak{p}}\|_0^2$ in Corollary 6.5 below.

Proposition 6.3. *The support* (5.1) *of ξ is the disjoint union*

$$H \begin{pmatrix} \varpi^{n-1} & & & \\ & \ddots & & \\ & & \varpi & \\ & & & 1 \end{pmatrix} K_1(\mathfrak{p}^{n+1}) = \bigcup_\alpha \bigcup_{y_1,\dots,y_{n-1}} H \begin{pmatrix} \varpi^{n-1} y_{n-1} & & & \\ & \ddots & & \\ & & \varpi y_1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix},$$

where $\alpha \in \mathrm{GL}_{n-1}(\mathfrak{o})$ runs through a set of representatives for $X_{n-1} \backslash \mathrm{GL}_{n-1}(\mathfrak{o})$ with

(6.7) $$X_n = \begin{pmatrix} \mathfrak{o}^* & \mathfrak{o} & \cdots & & \mathfrak{o} \\ \mathfrak{p}^2 & \mathfrak{o}^* & \cdots & & \mathfrak{o} \\ \mathfrak{p}^3 & \mathfrak{p}^2 & \cdots & & \mathfrak{o} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{p}^n & \mathfrak{p}^{n-1} & \cdots & \mathfrak{p}^2 & \mathfrak{o}^* \end{pmatrix},$$

and $y_1, \dots, y_{n-1} \in \mathfrak{o}^*$ run through a set of representatives for $\mathfrak{o}^*/(1 + \mathfrak{p}) \cong \Bbbk^*$.

Proof. It is easy to check that for $k \in K_1(\mathfrak{p}^{n+1})$,

$$\begin{pmatrix} \varpi^{n-1} & & & \\ & \ddots & & \\ & & \varpi & \\ & & & 1 \end{pmatrix} k \begin{pmatrix} \varpi^{n-1} & & & \\ & \ddots & & \\ & & \varpi & \\ & & & 1 \end{pmatrix}^{-1} \in H \begin{pmatrix} \mathfrak{o}^* & & & \\ & \ddots & & \\ & & \mathfrak{o}^* & \\ & & & 1 \end{pmatrix}$$

if and only if $k = \begin{pmatrix} Y & * \\ * & * \end{pmatrix}$ for $Y \in X_{n-1}$. Let $h \in H$ and $k \in K_1(\mathfrak{p}^{n+1})$. We can write $k = \begin{pmatrix} Y & * \\ * & * \end{pmatrix} \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix}$ for $Y \in X_{n-1}$ and $\alpha \in X_{n-1} \backslash \mathrm{GL}_{n-1}(\mathfrak{o})$. Then

conjugating $\left(\begin{smallmatrix} Y & * \\ * & * \end{smallmatrix}\right)$ as above, we obtain

$$h \begin{pmatrix} \varpi^{n-1} & & \\ & \ddots & \\ & & \varpi \\ & & & 1 \end{pmatrix} k = h' \begin{pmatrix} \varpi^{n-1} y_{n-1} & & \\ & \ddots & \\ & & \varpi y_1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix},$$

where $h' \in H$ and $y_1, \ldots, y_{n-1} \in \mathfrak{o}^*$. To prove uniqueness, suppose

$$(6.8) \quad h_1 \begin{pmatrix} \varpi^{n-1} y_{n-1} & & \\ & \ddots & \\ & & \varpi y_1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \\ & 1 \end{pmatrix} = h_2 \begin{pmatrix} \varpi^{n-1} z_{n-1} & & \\ & \ddots & \\ & & \varpi z_1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 & \\ & 1 \end{pmatrix}.$$

This yields an equality of the form $\left(\begin{smallmatrix} X & * \\ * & * \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha_1 & \\ & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \alpha_2 & \\ & 1 \end{smallmatrix}\right)$, where $X \in X_{n-1}$. Hence $\alpha_1 = \alpha_2$. Cancelling the α-matrices in (6.8) we see that h_1 and h_2 have the same last column. Multiplying each side by the same central element, we can assume that the entry in the lower right corner is 1, and hence that $h_1, h_2 \in K'$. Now multiplying both sides on the right by $\mathrm{diag}(\varpi^{n-1} z_{n-1}, \ldots, \varpi z_1, 1)^{-1}$ and equating the main diagonal entries on each side, we see that $y_j \equiv z_j \bmod \mathfrak{p}$, as needed. ∎

We can make the above explicit as follows.

Proposition 6.4. *Let X_n be the subgroup of K defined in* (6.7). *Then*

$$(6.9) \qquad [K : X_n] = q^{\frac{n(n-1)(n+1)}{6}} \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}{(q - 1)^n}.$$

Explicitly, if $B(\Bbbk) \subset \mathrm{GL}_n(\Bbbk)$ is the subgroup of upper triangular matrices, and $\overline{\Gamma} \subset \mathrm{GL}_n(\Bbbk)$ is a set of coset representatives for $B(\Bbbk) \backslash \mathrm{GL}_n(\Bbbk)$, obtained, e.g., from the Bruhat decomposition, and lifting bijectively to a set $\Gamma \subset K$, then a set of representatives for $X_n \backslash K$ is given by

$$(6.10) \qquad \left\{ \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ \delta_{2,1} & 1 & \cdots & & 0 \\ \delta_{3,1} & \delta_{3,2} & 1 & & 0 \\ \vdots & & \ddots & \ddots & 0 \\ \delta_{n,1} & \delta_{n,2} & \cdots & \delta_{n,n-1} & 1 \end{pmatrix} \gamma \;\middle|\; \delta_{ij} \in \mathfrak{p}/\mathfrak{p}^{i-j+1}, \gamma \in \Gamma \right\}.$$

Proof. Consider the containments

$$(6.11) \qquad X_n \subset \begin{pmatrix} \mathfrak{o}^* & \mathfrak{o} & \cdots & & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^* & \cdots & & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{p} & \cdots & & \mathfrak{o} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{p} & \mathfrak{p} & \cdots & \mathfrak{p} & \mathfrak{o}^* \end{pmatrix} \subset K.$$

Reducing modulo $\mathfrak{p}$ we see that the index of the right-hand containment is

$$[\mathrm{GL}_n(\Bbbk):B(\Bbbk)] = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}{(q-1)^n \, q^{\frac{n(n-1)}{2}}} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{(q-1)^n}.$$

Furthermore, one sees by induction that the δ_{ij}-matrices in (6.10) comprise a set of representatives for the quotient of the first containment in (6.11), or equivalently, for $(X_n \cap K')\backslash K'$. Counting along each subdiagonal, the number of such δ_{ij}-matrices is seen to be

$$(q)^{n-1}(q^2)^{n-2}\cdots(q^{n-1})^1 = q^{\sum_{i=1}^{n-1}(n-i)i} = q^{\frac{n(n-1)(n+1)}{6}}.$$

Multiplying these together, we obtain the index (6.9). $\blacksquare$

Corollary 6.5. *If Haar measure is normalized so that* $\mathrm{meas}(\overline{K}) = 1$*, then*

$$(6.12) \qquad \|\phi_{\mathfrak{p}}\|_0^2 = \frac{nq^{\frac{n(n-1)(n-2)}{6}}(q-1)}{q^n-1}.$$

Proof. Using Proposition 6.3, upon multiplying (6.9) (with $n-1$ in place of n) by $(q-1)^{n-1}$ we find that

$$|H\backslash\mathrm{Supp}(\phi_{\mathfrak{p}})| = q^{\frac{n(n-1)(n-2)}{6}}(q^{n-1}-1)(q^{n-2}-1)\cdots(q-1).$$

As a result, by (6.6) we have

$$(6.13) \qquad \|\phi_{\mathfrak{p}}\|_0^2 = n\,\mathrm{meas}(\overline{H})q^{\frac{n(n-1)(n-2)}{6}}(q^{n-1}-1)(q^{n-2}-1)\cdots(q-1).$$

To prove (6.12), under the normalization $\mathrm{meas}(\overline{K}) = 1$, we have

$$\mathrm{meas}(\overline{H}) = \mathrm{meas}(\overline{K'}) = [\overline{K}:\overline{K'}]^{-1}.$$

Consider $\overline{K(\mathfrak{p})} \subset \overline{K'} \subset \overline{K}$, where $K(\mathfrak{p}) = 1 + M_{n\times n}(\mathfrak{p})$. Taking the quotient by $\overline{K(\mathfrak{p})}$, we get $1 \subset N(\Bbbk) \subset \mathrm{PGL}_n(\Bbbk)$. Thus

$$\begin{aligned}
[\overline{K}:\overline{K'}] &= \frac{|\mathrm{PGL}_n(\Bbbk)|}{|N(\Bbbk)|} = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})}{(q-1)q^{n(n-1)/2}} \\
&= \frac{(q^n-1)(q^{n-1}-1)\cdots(q-1)}{q-1} \\
&= (q^n-1)(q^{n-1}-1)\cdots(q^2-1).
\end{aligned}$$

Equation (6.12) now follows. $\blacksquare$

7. Low Rank Examples

When $n = 2$, the support of ξ is equal to

$$H \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K_1(\mathfrak{p}^3) = \bigcup_{y \in \mathbf{k}^*} H \begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix},$$

so the new vector $\phi_\mathfrak{p}$ is supported on the disjoint union

$$(7.1) \qquad g_\chi H \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K_1(\mathfrak{p}^3) = \bigcup_{y \in \mathbf{k}^*} H g_\chi \begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix}.$$

When $n = 3$, we apply (6.10) (with $n = 2$) to see by Proposition 6.3 that $\phi_\mathfrak{p} = \xi$ is supported on the disjoint union

$$(7.2) \quad \bigcup_{y_1, y_2 \in \mathbf{k}^*} \left(\bigcup_{\delta \in \mathfrak{p}/\mathfrak{p}^2} H \begin{pmatrix} \varpi^2 y_2 & \\ & \varpi y_1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \\ \delta & 1 & \\ & & 1 \end{pmatrix} \cup \bigcup_{\tau \in \mathfrak{o}/\mathfrak{p}^2} H \begin{pmatrix} \varpi^2 y_2 & \\ & \varpi y_1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \\ 1 & \tau & \\ & & 1 \end{pmatrix} \right).$$

In the $n = 2$ case, we can compute the matrix coefficient as a function of the matrix entries of g.

Theorem 7.1. *Let $n = 2$, and let $\chi \leftrightarrow \langle 1, t \rangle$ be the affine generic character of K' determined by $t \in \mathbb{k}^*$ and the unitary central character $\omega : F^* \to \mathbf{C}^*$ trivial on $1 + \mathfrak{p}$. Fix $\zeta \in \mathbf{C}$ satisfying $\zeta^2 = \omega(t\varpi)$, and let σ_χ^ζ be the associated simple supercuspidal representation. Define the matrix coefficient*

$$f_\mathfrak{p}(g) = d_\chi \overline{\left\langle \sigma_\chi^\zeta(g) v, v \right\rangle},$$

where v is a new vector for σ_χ^ζ of norm 1, and d_χ is the formal degree relative to the Haar measure on $\overline{G}$ in which $\mathrm{meas}(\overline{K}) = 1$. Then for $z \in Z$,

$$(7.3)$$
$$f_\mathfrak{p}(zg) = \begin{cases} \dfrac{(q+1)}{2\omega(z)} \displaystyle\sum_{y \in \mathbf{k}^*} \psi(by + \tfrac{tc}{a} y^{-1}) & \text{if } g = \left(\begin{smallmatrix} a & b\varpi^{-1} \\ c\varpi^2 & d \end{smallmatrix} \right) \in \left(\begin{smallmatrix} \mathfrak{o}^* & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{smallmatrix} \right) \\[2.5ex] \dfrac{(q+1)\zeta}{2\omega(-zd)} \displaystyle\sum_{y \in \mathbf{k}^*} \overline{\omega(y)} \psi(\tfrac{c}{a} y + \tfrac{tb}{d} y^{-1}) & \text{if } g = \left(\begin{smallmatrix} c & d\varpi^{-2} \\ a\varpi & b \end{smallmatrix} \right) \in \left(\begin{smallmatrix} \mathfrak{o} & \varpi^{-2}\mathfrak{o}^* \\ \varpi\mathfrak{o}^* & \mathfrak{o} \end{smallmatrix} \right). \end{cases}$$

This expression determines $f_\mathfrak{p}$ since it vanishes outside the disjoint union

$$(7.4) \qquad Z \cdot \begin{pmatrix} \mathfrak{o}^* & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix} \bigcup Z \cdot \begin{pmatrix} \mathfrak{o} & \varpi^{-2}\mathfrak{o}^* \\ \varpi\mathfrak{o}^* & \mathfrak{o} \end{pmatrix}.$$

Remarks.

(1) The first case is a Kloosterman sum, and the second case is a twisted Kloosterman sum.

(2) The reason for the complex conjugate in the definition of $f_{\mathfrak{p}}$ is that then for any irreducible unitary representation π of $\mathrm{GL}_2(F)$, the operator $\pi(f_{\mathfrak{p}})$ is either the orthogonal projection onto v (if $\pi = \sigma_\chi^\zeta$) or 0 (if $\pi \not\cong \sigma_\chi^\zeta$). See Corollary 10.29 of [KL1].

Proof. Using the model of σ_χ^ζ on A_0, we will compute $I(g) = d_\chi \left\langle \sigma_\chi^\zeta(g)\phi_{\mathfrak{p}}, \phi_{\mathfrak{p}} \right\rangle_0$, where $\phi_{\mathfrak{p}}$ is the new vector given in Corollary 5.2. Clearly $I(g)$ and $f_{\mathfrak{p}}(g)$ have the same support. First suppose that $g \in E$. Then (6.6), together with (7.1), gives

$$I(g) = \sum_{y \in \mathbf{k}^*} \phi_{\mathfrak{p}}(g_\chi \begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix} g) = \sum_{y \in \mathbf{k}^*} \xi(\begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix} g)$$

(note that the term $\overline{\phi_{\mathfrak{p}}(x)}$ in (6.6) is equal to 1 when $x \in g_\chi \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} K_1(\mathfrak{p}^3)$ as is the case here). If g belongs to the support, then for some y,

$$g \in \bigcup_{z \in \mathfrak{o}^*/(1+\mathfrak{p})} \begin{pmatrix} \varpi^{-1}y^{-1} & \\ & 1 \end{pmatrix} H \begin{pmatrix} \varpi z & \\ & 1 \end{pmatrix} = \bigcup_{z \in \mathfrak{o}^*/(1+\mathfrak{p})} \begin{pmatrix} \varpi^{-1} & \\ & 1 \end{pmatrix} H \begin{pmatrix} \varpi z & \\ & 1 \end{pmatrix}.$$

It is easy to show that this set coincides with the left-hand set in (7.4). Let $g = \begin{pmatrix} a & b\varpi^{-1} \\ c\varpi^2 & d \end{pmatrix} \in \begin{pmatrix} \mathfrak{o}^* & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix}$. Noting that

$$\xi(\begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix} \begin{pmatrix} a & b\varpi^{-1} \\ c\varpi^2 & d \end{pmatrix}) = \xi(\begin{pmatrix} 1 & by \\ \varpi c(ay)^{-1} & d \end{pmatrix} \begin{pmatrix} \varpi ay & \\ & 1 \end{pmatrix}) = \psi(by + tc(ay)^{-1}),$$

and then replacing y by $-y$, we find

$$I(g) = \sum_{y \in \mathbf{k}^*} \psi(by + tc(ay)^{-1}) = \sum_{y \in \mathbf{k}^*} \psi(-by - tc(ay)^{-1})$$

Similarly, if $g \in Eg_\chi$, then (6.6) and (7.1) with $r = 1$ give

$$I(g) = \zeta \sum_{y \in \mathbf{k}^*} \phi_{\mathfrak{p}}(g_\chi^{-1} g_\chi \begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix} g) = \zeta \sum_{y \in \mathbf{k}^*} \xi(g_\chi^{-1} \begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix} g).$$

If g belongs to the support, then for some y

(7.5) $$g \in \bigcup_{z \in \mathfrak{o}^*/(1+\mathfrak{p})} \begin{pmatrix} \varpi^{-1}y^{-1} & \\ & 1 \end{pmatrix} g_\chi H \begin{pmatrix} \varpi z & \\ & 1 \end{pmatrix}$$

For any $\left(\begin{smallmatrix} m & s \\ \varpi x & u \end{smallmatrix}\right) \in K'$, we have

$$(7.6) \quad \begin{pmatrix} \varpi^{-1}y^{-1} & \\ & 1 \end{pmatrix}\begin{pmatrix} & t \\ \varpi & \end{pmatrix}\begin{pmatrix} m & s \\ \varpi x & u \end{pmatrix}\begin{pmatrix} \varpi z & \\ & 1 \end{pmatrix} = \begin{pmatrix} \varpi txy^{-1}z & tu\varpi^{-1}y^{-1} \\ \varpi^2 mz & s\varpi \end{pmatrix}.$$

From this computation, we see that the union (over y) of the sets (7.5) coincides with the right-hand set in (7.4). Therefore if $g = \left(\begin{smallmatrix} c & d\varpi^{-2} \\ a\varpi & b \end{smallmatrix}\right) \in \left(\begin{smallmatrix} \mathfrak{o} & \mathfrak{o}^*\varpi^{-2} \\ \mathfrak{o}^*\varpi & \mathfrak{o} \end{smallmatrix}\right)$,

$$
\begin{aligned}
I(g) &= \zeta \sum_{y \in \mathbf{k}^*} \xi\left(\begin{pmatrix} & \varpi^{-1} \\ t^{-1} & \end{pmatrix}\begin{pmatrix} \varpi y & \\ & 1 \end{pmatrix}\begin{pmatrix} c & d\varpi^{-2} \\ a\varpi & b \end{pmatrix}\right) \\
&= \zeta \sum_{y \in \mathbf{k}^*} \xi\left(\begin{pmatrix} a & b\varpi^{-1} \\ cyt^{-1}\varpi & dyt^{-1}\varpi^{-1} \end{pmatrix}\right) \\
&= \frac{\zeta}{\omega(\varpi)} \sum_{y \in \mathbf{k}^*} \xi\left(\begin{pmatrix} a & b \\ cyt^{-1}\varpi & dyt^{-1} \end{pmatrix}\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}\right) \\
&= \frac{\omega(t)}{\zeta} \sum_{y \in \mathbf{k}^*} \xi\left(dyt^{-1}\begin{pmatrix} ad^{-1}y^{-1}t & bd^{-1}y^{-1}t \\ cd^{-1}\varpi & 1 \end{pmatrix}\begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}\right) \\
&= \frac{\omega(d)}{\zeta} \sum_{y \in \mathbf{k}^*} \omega(y)\xi\left(\begin{pmatrix} 1 & bd^{-1}y^{-1}t \\ ca^{-1}t^{-1}y\varpi & 1 \end{pmatrix}\begin{pmatrix} \frac{at}{dy}\varpi & \\ & 1 \end{pmatrix}\right) \\
&= \frac{\omega(d)}{\zeta} \sum_{y \in \mathbf{k}^*} \omega(y)\psi(\tfrac{tb}{d}y^{-1} + \tfrac{c}{a}y) = \frac{\omega(-d)}{\zeta} \sum_{y \in \mathbf{k}^*} \omega(y)\psi(-\tfrac{tb}{d}y^{-1} - \tfrac{c}{a}y).
\end{aligned}
$$

The proposition now follows upon taking complex conjugates, and multiplying by $\|\phi_{\mathfrak{p}}\|^{-2} = \frac{(q+1)}{2}$ (cf. (6.12)). ∎

Proposition 7.2. *Let π be an irreducible admissible representation of $\mathrm{GL}_2(F)$ of conductor $\mathfrak{p}^3$ and central character ω trivial on $1+\mathfrak{p}$. Then π is a simple supercuspidal representation.*

Proof. We first prove that π is supercuspidal. Generally, if $\mathfrak{c}(\pi)$ is the conductor of π, then

$$\mathfrak{c}(\pi) = \begin{cases} \mathfrak{c}(\chi_1)\mathfrak{c}(\chi_2) & \text{if } \pi = \pi(\chi_1,\chi_2) \text{ (principal series)} \\ \mathfrak{p} & \text{if } \pi = \mathrm{St}\otimes\chi \text{ (unramified twist of Steinberg)} \\ \mathfrak{c}(\chi)^2 & \text{if } \pi = \mathrm{St}\otimes\chi \text{ (ramified twist of Steinberg)} \end{cases}$$

(e.g. see the end of §1 of [S]). Clearly π cannot be a twist of the Steinberg representation when $\mathfrak{c}(\pi) = \mathfrak{p}^3$. If π is principal series, then $\mathfrak{p}^3 = \mathfrak{c}(\pi) = \mathfrak{c}(\chi)\mathfrak{c}(\chi^{-1}\omega)$ for some character χ which is necessarily ramified. If $\mathfrak{c}(\chi) = \mathfrak{p}$, then $\mathfrak{c}(\chi^{-1}\omega) = \mathfrak{p}$ or 1, so $\mathfrak{c}(\pi)|\mathfrak{p}^2$, a contradiction. If $\mathfrak{p}^2|\mathfrak{c}(\chi)$, then $\mathfrak{c}(\chi^{-1}\omega) = \mathfrak{c}(\chi)$, so $\mathfrak{p}^4|\mathfrak{c}(\pi)$, another contradiction. Hence π is supercuspidal, and therefore simple supercuspidal by Corollary 5.3. ∎

8. Proof of Theorem 5.1

In this section we prove Theorem 5.1. In fact, we will compute $\pi_\chi^{K_1(\mathfrak{p}^m)}$ for any $m \geq 1$. This space is spanned by functions supported on double cosets $HgK_1(\mathfrak{p}^m)$. Our method is a direct calculation:

(A) Produce an explicit set of matrices containing a full set of representatives for $H\backslash G/K_1(\mathfrak{p}^m)$.

(B) Determine which of these double cosets can support well-defined functions in $\pi_\chi^{K_1(\mathfrak{p}^m)}$.

The final result is stated in Theorem 8.10 below.

8.1. Notation for the proof

In this section, χ is the affine generic character corresponding to $\langle t_1, t_2, \ldots, t_n \rangle \in (\Bbbk^*)^n$. For $S \subset F$, let $M_{n \times n}(S)$ denote the set of $n \times n$ matrices with entries in S. Let $B_n(\mathfrak{o})$ denote the set of upper triangular matrices in $\mathrm{GL}_n(\mathfrak{o})$.

As before, E_{ij} denotes the $n \times n$ matrix whose only non-zero entry is a 1 in the i-th row and j-th column. Lastly, let

$$K'_{n-1} \subset \mathrm{GL}_{n-1}(\mathfrak{o})$$

be the analog of K', of dimension $(n-1) \times (n-1)$.

8.2. $K_1(\mathfrak{p}^m)$-invariant elements of π_χ

For $g \in G$, let

$$[g] = HgK_1(\mathfrak{p}^m).$$

Suppose f is a $K_1(\mathfrak{p}^m)$-invariant function in the space of π_χ. The value $f(g)$ determines the values of f on $HgK_1(\mathfrak{p}^m)$ via $f(hgk) = \chi(h)f(g)$. Therefore the function f is determined by its values on any set of representatives for the double quotient $H\backslash G/K_1(\mathfrak{p}^m)$. We say that g and the double coset $[g]$ are **relevant for** $K_1(\mathfrak{p}^m)$ (or simply **relevant**, if m is clear from the context) if there exists such f for which $f(g) \neq 0$.

Proposition 8.1. *Given $m \geq 1$, an element $g \in G$ is relevant if and only if*

(8.1) $$\chi \text{ is trivial on } gK_1(\mathfrak{p}^m)g^{-1} \cap K'.$$

This condition is independent of the choice of representative g for the double coset $[g]$. In particular, if g satisfies condition (8.1), there exists a $K_1(\mathfrak{p}^m)$-invariant function in π_χ, unique up to multiples, whose support is $[g]$. We let f_g denote the unique such function satisfying $f_g(g) = 1$.

If $[g_1], [g_2], [g_3], \ldots$ is a list of all relevant double cosets (noting that $H\backslash G/K_1(\mathfrak{p}^m)$ is countable since G is separable), then $\{f_{g_1}, f_{g_2}, \ldots\}$ is a basis for $\pi_\chi^{K_1(\mathfrak{p}^m)}$.

Proof. Suppose $f(g) \neq 0$ for some $K_1(\mathfrak{p}^m)$-invariant function f in π_χ. Given any $h \in gK_1(\mathfrak{p}^m)g^{-1} \cap K' \subset H$, we have $hg = gk$ for some $k \in K_1(\mathfrak{p}^m)$, so $f(g) = f(gk) = f(hg) = \chi(h)f(g)$. Because $f(g) \neq 0$, $\chi(h) = 1$. Thus g satisfies (8.1). Obviously this condition is independent of the choice of representative for $HgK_1(\mathfrak{p}^m)$.

Conversely, suppose g satisfies condition (8.1). Define

$$f(x) = \begin{cases} \chi(h) & \text{if } x = hgk \text{ for some } h \in H, k \in K_1(\mathfrak{p}^m), \\ 0 & \text{otherwise.} \end{cases}$$

We need to show that the function is well-defined, i.e. $\chi(h_1) = \chi(h_2)$ whenever $h_1gk_1 = h_2gk_2$ with $h_1, h_2 \in H$ and $k_1, k_2 \in K_1(\mathfrak{p}^m)$. Write $h_i = s_i k_i'$ with $s_i \in F^*$ and $k_i' \in K'$ for $i = 1, 2$. Then

$$h_2^{-1}h_1 = k_2'^{-1}s_2^{-1}s_1 k_1' = gk_2 k_1^{-1}g^{-1}.$$

Taking determinants, we see that $s_2^{-1}s_1 \in \mathfrak{o}^*$. The characteristic polynomial of the middle matrix modulo $\mathfrak{p}$ is $(X - s_2^{-1}s_1)^n$. Because $k_2 k_1^{-1} \in K_1(\mathfrak{p}^m)$, the characteristic polynomial of $gk_2 k_1^{-1}g^{-1}$ modulo $\mathfrak{p}$ has a factor of $(X-1)$. Thus $s_2^{-1}s_1 \equiv 1 \pmod{\mathfrak{p}}$. Therefore $h_2^{-1}h_1 \in gK_1(\mathfrak{p}^m)g^{-1} \cap K'$, so by (8.1), $\chi(h_2^{-1}h_1) = 1$ and hence $\chi(h_1) = \chi(h_2)$, as required.

Now let $\{[g_1], [g_2], \ldots\}$ be the set of all relevant double cosets. Because $f_{g_1}, f_{g_2}, \ldots$ have pairwise disjoint supports, they form a linearly independent set. If f is any $K_1(\mathfrak{p}^m)$-invariant function, then the support of f is a finite disjoint union of relevant double cosets $\bigcup_{i=1}^\ell Hg_{j_i}K_1(\mathfrak{p}^m)$, so $f = \sum_{i=1}^\ell f(g_{j_i})f_{g_{j_i}}$. This proves the final assertion. ∎

8.3. Representatives for $H\backslash G/K_1(\mathfrak{p}^m)$

We say that an element $g \in G$ is **primitive** if $g \in M_{n\times n}(\mathfrak{o})$ and some entry of g is a unit. Let $P_{n\times n}(\mathfrak{o})$ be the set of primitive matrices. It is easy to see that every class $[g]$ contains a primitive element.

Lemma 8.2. *Let $g \in G$ be primitive. Then*

$$[g] \cap M_{n\times n}(\mathfrak{o}) = (\mathfrak{o} \cap F^*) \cdot K'gK_1(\mathfrak{p}^m),$$

$$[g] \cap P_{n\times n}(\mathfrak{o}) = \mathfrak{o}^* \cdot K'gK_1(\mathfrak{p}^m).$$

Proof. Suppose $h \in [g] \cap M_{n\times n}(\mathfrak{o})$. Then $h = zk'gk$ for some $z \in F^*$, $k' \in K'$ and $k \in K_1(\mathfrak{p}^m)$. Because k' and k are in $\mathrm{GL}_n(\mathfrak{o})$, $k'gk$ is also primitive. It follows that $z \in \mathfrak{o} \cap F^*$. The second claim can be proven similarly. ∎

We say that a matrix $g \in M_{n\times n}(\mathfrak{o})$ is **divisible by** g_χ if $g_\chi^{-1}g \in M_{n\times n}(\mathfrak{o})$, or equivalently, if all entries of the bottom row of g belong to $\mathfrak{p}$.

Proposition 8.3. *Let $g \in G$ be primitive. Suppose $h \in [g] \cap P_{n \times n}(\mathfrak{o})$. Then g is divisible by g_χ if and only if h is divisible by g_χ.*

Proof.   By the above lemma, $h = zk'gk$ for $z \in \mathfrak{o}^*$, $k' \in K'$ and $k \in K_1(\mathfrak{p}^m)$. Since $k' \in K'$, it is clear that if each entry in the last row of g belongs to $\mathfrak{p}$, then the same is true of h. The converse is also easy to see, using $g = z^{-1}k'^{-1}hk^{-1}$.   ∎

We say that a double coset is **divisible by** g_χ if it contains a primitive element that is divisible by g_χ. By the above proposition, this is equivalent to saying that every primitive element in the double coset is divisible by g_χ.

Proposition 8.4. *For any $g \in G$, there exists an integer $0 \le r < n$ and a double coset $[h]$ not divisible by g_χ such that $[g] = g_\chi^r[h] = [g_\chi^r h]$.*

Proof.   Since $[g]$ contains a primitive element, we may assume that g is primitive. Let $r \ge 0$ be the largest integer for which $g_\chi^{-r}g \in M_{n \times n}(\mathfrak{o})$. Let $h = g_\chi^{-r}g$. Then h is not divisible by g_χ. If $r \ge n$, then because $g_\chi^n = \varpi \frac{t_n^{n-1}}{t_1 \cdots t_{n-1}} I_n$, the matrix $g = g_\chi^r h = g_\chi^n g_\chi^{-n} h$ is not primitive, which is a contradiction. By Lemma 3.2, $[g_\chi^r h] = g_\chi^r[h]$.   ∎

Proposition 8.5. *An element $g \in G$ is relevant if and only if $g_\chi^{-1}g$ is relevant. Likewise, for $r \ge 0$, $[g]$ is relevant if and only if $[g_\chi^{-r}g]$ is relevant.*

Proof.   This is immediate from Lemma 3.2.   ∎

For $g \in G$, let $\widetilde{g}$ denote the $(n-1) \times (n-1)$ submatrix of g formed by removing the last row and last column of g.

Lemma 8.6. *Given $g \in G$, there exists an element $h \in K'g$ with $\det \widetilde{h} \ne 0$.*

Proof.   Let g' be the matrix obtained by deleting the last column of g. Because the rank of g is n, g' has rank $n-1$. If the first $n-1$ rows of g' are linearly independent, then we can take $h = g$. Otherwise, there exists an index $i \le n-1$ such that the rows of g' other than the i-th row are linearly independent. Then we can take $h = (I_n + E_{in})g$.   ∎

Given $g \in G$, there exists $z \in Z$ such that $zg \in M_{n \times n}(\mathfrak{o})$. By Lemma 8.6, there is an element $h \in K'zg$ such that $\det \widetilde{h} \ne 0$. Therefore the set

$$\{h \in HgK_1(\mathfrak{p}^m) \bigcap M_{n \times n}(\mathfrak{o}) \mid \det \widetilde{h} \ne 0\}$$

is non-empty. We let

(8.2) $$\mu_g = \mu_{[g]} \ge 0$$

denote the minimum of $\mathrm{ord}_{\mathfrak{p}}(\det \widetilde{h})$ as h ranges through the above set.

Lemma 8.7. *Let $g \in G$. Then if $\det \widetilde{g} \neq 0$, there exists $h \in K'gK_1(\mathfrak{p}^m)$ such that $\widetilde{h}$ is diagonal and $\mathrm{ord}_{\mathfrak{p}}(\det \widetilde{g}) = \mathrm{ord}_{\mathfrak{p}}(\det \widetilde{h})$.*

Proof. By the Cartan decomposition, there exist $k_1, k_2 \in \mathrm{GL}_{n-1}(\mathfrak{o})$ and a diagonal matrix

$$\delta = \begin{pmatrix} \varpi^{a_1} & & & \\ & \varpi^{a_2} & & \\ & & \ddots & \\ & & & \varpi^{a_{n-1}} \end{pmatrix}$$

where $a_1 \leq a_2 \leq \cdots \leq a_{n-1}$ are integers, such that $\widetilde{g} = k_1 \delta k_2$. By the Bruhat decomposition of $\mathrm{GL}_{n-1}(\mathfrak{o}/\mathfrak{p})$, there exist matrices $k' \in K'_{n-1}$, $b \in B_{n-1}(\mathfrak{o})$ and a Weyl element $w \in \mathcal{W}_{n-1}$ such that

$$k_1 \equiv k'wb \pmod{\mathfrak{p}}.$$

Let $k'' = k_1(wb)^{-1} \equiv k' \pmod{\mathfrak{p}}$. Obviously $k'' \in K'_{n-1}$. Then

$$\widetilde{g} = k_1 \delta k_2 = k''wb\delta k_2 = k''(w\delta w^{-1})w(\delta^{-1}b\delta)k_2.$$

A simple calculation shows that $\delta^{-1}b\delta$ has entries in $\mathfrak{o}$. Because $\det \delta^{-1}b\delta = \det b$ is a unit, it is in $\mathrm{GL}_{n-1}(\mathfrak{o})$. Let $k = w(\delta^{-1}b\delta)k_2 \in \mathrm{GL}_{n-1}(\mathfrak{o})$. Note also that $\delta' = w\delta w^{-1}$ is diagonal. It follows that if we let

$$h = \begin{pmatrix} k'' & \\ & 1 \end{pmatrix}^{-1} g \begin{pmatrix} k & \\ & 1 \end{pmatrix}^{-1} \in K'gK_1(\mathfrak{p}^m),$$

then $\widetilde{h} = k''^{-1}\widetilde{g}k^{-1} = \delta'$ is diagonal, and $\mathrm{ord}_{\mathfrak{p}}(\det \widetilde{h}) = \mathrm{ord}_{\mathfrak{p}}(\det \widetilde{g})$. ∎

Theorem 8.8. *Let $[g]$ be a double coset not divisible by g_χ. Let $\mathcal{R}$ containing 0 be a fixed complete set of representatives in $\mathfrak{o}$ for $\mathfrak{o}/\mathfrak{p}$. Then there exists an element in $[g]$ of the following form:*

$$(8.3) \qquad h = \begin{pmatrix} \varpi^{b_1} & 0 & \cdots & 0 & v_1 \\ 0 & \varpi^{b_2} & \cdots & 0 & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \varpi^{b_{n-1}} & v_{n-1} \\ \varpi^{b_1}u_1 & \varpi^{b_2}u_2 & \cdots & \varpi^{b_{n-1}}u_{n-1} & w \end{pmatrix}.$$

Here for each i, $b_i \geq 0$ is an integer, the elements u_i, v_i, w belong to $\mathfrak{o}$, and the following conditions are satisfied:

   (a) h *is primitive.*

   (b) $\mathrm{ord}_{\mathfrak{p}}(\det \widetilde{h}) = b_1 + b_2 + \cdots + b_{n-1} = \mu_g$ *as defined in (8.2).*

   (c) $u_i \in \mathcal{R}$ *for $i = 1, 2, \ldots, n-1$. In particular, if u_i is not a unit, then $u_i = 0$.*

(d) *Suppose $i < j \le n - 1$, $b_i \le b_j$ and $u_i \ne 0$. Then $u_j = 0$.*

(e) $w \in \mathfrak{o}^*$.

Remark. The correspondence between the specified set of double cosets and the matrices (8.3) is not one-to-one. For example, given $1 \le i < n$ and $a \in \mathfrak{o}$, we can replace h by $h(I_n + a\varpi E_{in}) \in hK_1(\mathfrak{p}^m)$. The new matrix is still of the form (8.3), with v_i replaced by $v_i + a\varpi^{b_i+1}$ and w replaced by $w + a\varpi^{b_i+1} \in \mathfrak{o}^*$. Nevertheless, it is sufficient for our purpose.

Proof. Let $h \in [g] \cap M_{n \times n}(\mathfrak{o})$ with $\mathrm{ord}_{\mathfrak{p}}(\det \tilde{h}) = \mu_g$ (cf. (8.2)). Note that h is primitive, since otherwise we could replace it with $\varpi^{-1}h$ and get a smaller valuation. Furthermore, applying Lemma 8.7 to h, we can assume that $\tilde{h}$ is diagonal, and therefore

$$(8.4) \qquad h = \begin{pmatrix} \varpi^{b_1} & & & & v_1 \\ & \varpi^{b_2} & & & v_2 \\ & & \ddots & & \vdots \\ & & & \varpi^{b_{n-1}} & v_{n-1} \\ \beta_1 & \beta_2 & \cdots & \beta_{n-1} & w \end{pmatrix},$$

where $b_1, \ldots, b_{n-1} \ge 0$ and $v_1, \ldots, v_{n-1}, \beta_1, \ldots, \beta_{n-1}, w \in \mathfrak{o}$. Lemma 8.7 also allows us to assume that property (b) holds.

Adding the last row of h to the i-th row, we obtain

$$h' = (1 + E_{in})h = \begin{pmatrix} \varpi^{b_1} & & & & & & v_1 \\ & \ddots & & & & & \vdots \\ \beta_1 & \cdots & \varpi^{b_i} + \beta_i & \cdots & \beta_{n-1} & & v_i + w \\ & & & \ddots & & & \vdots \\ & & & & \varpi^{b_{n-1}} & & v_{n-1} \\ \beta_1 & \cdots & \beta_i & \cdots & \beta_{n-1} & & w \end{pmatrix}.$$

This is an element of $K'h \subset [g] \cap M_{n \times n}(\mathfrak{o})$, with

$$\det \tilde{h}' = (\varpi^{b_i} + \beta_i) \prod_{j \ne i} \varpi^j.$$

By the minimality of $\mathrm{ord}_{\mathfrak{p}}(\det \tilde{h})$, we must have $\mathrm{ord}_{\mathfrak{p}}(\varpi^{b_i} + \beta_i) \ge b_i$. This means that $\beta_i = \varpi^{b_i} u_i$ for some $u_i \in \mathfrak{o}$.

To prove (c), let $a \in \mathfrak{o}$ and $0 < i < n$. Adding the $a\varpi$-multiple of the i-th row of h to the bottom row, we obtain $h' = (I_n + a\varpi E_{ni})h \in K'h \subset [g] \cap M_{n \times n}(\mathfrak{o})$. Note that h' is of the form (8.4) with last row equal to

$$\begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{i-1} & \beta_i + a\varpi^{b_i+1} & \beta_{i+1} & \cdots & \beta_{n-1} & w + a\varpi v_i \end{pmatrix}.$$

Because $\beta_i + a\varpi^{b_i+1} = \varpi^{b_i}(u_i + a\varpi)$ and a is arbitrary, we can therefore assume that $u_i \in \mathcal{R}$. This proves (c).

To prove (d), suppose $i < j \leq n-1$, $b_i \leq b_j$ and $u_i \neq 0$. Because $u_i \in \mathcal{R}$ by (c), u_i is a unit. Let $a \in \mathfrak{o}$. Adding the a-multiple of the j-th row to the i-th row, then subtracting the $a\varpi^{b_j-b_i}$-multiple of the i-th column from the j-th column, we obtain

$$h' = (I_n + aE_{ij})h(I_n - a\varpi^{b_j-b_i}E_{ij}) \in [g] \cap M_{n\times n}(\mathfrak{o}),$$

where $\tilde{h}' = \tilde{h}$, and the last row of h' is

$$\begin{pmatrix} \beta_1 & \cdots & \beta_{j-1} & \beta_j - a\varpi^{b_j-b_i}\beta_i & \beta_{j+1} & \cdots & \beta_{n-1} & w \end{pmatrix}.$$

Now

$$\beta_j - a\varpi^{b_j-b_i}\beta_i = \varpi^{b_j}(u_j - au_i).$$

Because u_i is a unit, we can take $a = u_j/u_i$, and therefore replacing h by h' we can assume $u_j = 0$. This proves (d).

To prove (e), suppose $w \in \mathfrak{p}$. Because $[g]$ is not divisible by g_χ, $\beta_i \in \mathfrak{o}^*$ for some $i < n$. Adding the i-th column of h to the last column, we obtain

$$h' = h(I_n + E_{in}) \in hK_1(\mathfrak{p}^m) \subset [g] \cap M_{n\times n}(\mathfrak{o}).$$

The submatrix $\tilde{h}'$ is $\tilde{h}$. Its lower right corner entry is $w + \beta_i \in \mathfrak{o}^*$. Replacing h by h', we can assume that w is a unit. ∎

Corollary 8.9. *Every double coset in $H\backslash G/K_1(\mathfrak{p}^m)$ contains an element of the form $g_\chi^r h$ for $0 \leq r < n$ and h of the form* (8.3).

Proof. This is immediate from the above theorem and Proposition 8.4. ∎

8.4. $K_1(\mathfrak{p}^m)$-invariant functions

In this section, we will describe all relevant double cosets. The end result is the following theorem, of which Theorem 5.1 is an immediate corollary.

Theorem 8.10. *Given $m \geq 1$, a double coset $[g]$ is relevant for $K_1(\mathfrak{p}^m)$ if and only if $[g] = [g_\chi^a \delta]$ for some $0 \leq a \leq n-1$ and*

$$(8.5) \qquad \delta = \begin{pmatrix} \varpi^{b_1} & & \\ & \ddots & \\ & & \varpi^{b_n} \end{pmatrix}, \quad m-1 > b_1 > b_2 > \cdots > b_n = 0.$$

Consequently, in the notation of (4.2) *and Proposition 8.1, a basis for $\pi_\chi^{K_1(\mathfrak{p}^m)}$ is given by*

$$\{L^a f_\delta \mid 0 \leq a \leq n-1, \delta \text{ as in } (8.5)\}.$$

Corollary 8.11. *If $\pi_\chi^{K_1(\mathfrak{p}^m)}$ is nonzero, then $m \geq n + 1$.*

Proof. Suppose $f \in \pi_\chi^{K_1(\mathfrak{p}^m)}$ is non-zero and $f(g) \neq 0$. Then by Proposition 8.1, $[g]$ is relevant. By Theorem 8.10, we can assume that $g = g_\chi^a \delta$ as above. Because $m - 1 > b_1 > b_2 > \cdots > b_n = 0$, we have $m - 1 > n - 1$, i.e. $m \geq n + 1$. ∎

Suppose g is relevant and not divisible by g_χ. Without loss of generality, we can assume that g is given by (8.3). Let

$$d = w - \sum_{i=1}^{n-1} u_i v_i.$$

Then from the second-to-last row cofactor expansion we find inductively that

$$\det g = d \varpi^{b_1 + \cdots + b_{n-1}}.$$

In particular, $d \neq 0$. We will prove Theorem 8.10 by treating the two cases:

(1) $u_i \neq 0$ for some i,

(2) $u_1 = u_2 = \cdots = u_{n-1} = 0$.

8.4.1. Case (1): $u_i \neq 0$ for some i.

Let $\ell \geq 1$ be the minimal index such that $u_\ell \neq 0$, i.e., $u_1 = u_2 = \cdots = u_{\ell-1} = 0$, $u_\ell \neq 0$. By Theorem 8.8 (c), u_ℓ is a unit. Subtracting the u_ℓ^{-1}-multiple of the last row of g from the ℓ-th row, and then adding the $u_i u_\ell^{-1}$-multiple of the i-th row to the ℓ-th row for $i = \ell + 1, \ldots, n - 1$, we obtain:

$$\left(\prod_{i=\ell+1}^{n-1} (I_n + u_i u_\ell^{-1} E_{\ell i}) \right) (I_n - u_\ell^{-1} E_{\ell n}) g$$

$$= \begin{pmatrix} \varpi^{b_1} & & & & & & & v_1 \\ & \ddots & & & & & & \vdots \\ & & \varpi^{b_{\ell-1}} & & & & & v_{\ell-1} \\ & & & 0 & & & & -\frac{d}{u_\ell} \\ & & & & \varpi^{b_{\ell+1}} & & & v_{\ell+1} \\ & & & & & \ddots & & \vdots \\ & & & & & & \varpi^{b_{n-1}} & v_{n-1} \\ 0 & \cdots & 0 & \varpi^{b_\ell} u_\ell & \varpi^{b_{\ell+1}} u_{\ell+1} & \cdots & \varpi^{b_{n-1}} u_{n-1} & w \end{pmatrix} \in K'g \subset [g].$$

The (ℓ, n) entry is obtained by the fact that $d = w - \sum_{i=1}^{n-1} u_i v_i = w - \sum_{i=\ell}^{n-1} u_i v_i$. Therefore, we can replace g by the above matrix. From now on, we assume that g is the above matrix. We are going to show that g is not relevant, i.e. it does not satisfy (8.1). Suppose to the contrary that g satisfies (8.1). We will obtain a contradiction by establishing the following:

1. $v_1, v_2, \ldots, v_{\ell-1}$ can be assumed to be 0.

2. Either $\varpi | u_i$ or $\varpi | v_i$, i.e., $\varpi | u_i v_i$, for $i = \ell+1, \ldots, n-1$.

3. $\varpi | (w - \sum_{i=\ell+1}^{n-1} u_i v_i)$. Therefore $\varpi | w$ by the previous result. This contradicts the fact that w is a unit (cf. Theorem 8.8).

It is easy to verify that

$$
g^{-1} = \begin{pmatrix}
\varpi^{-b_1} & & & \frac{\varpi^{-b_1} u_\ell v_1}{d} & & & \\
& \ddots & & \vdots & & & \\
& & \varpi^{-b_{\ell-1}} & \frac{\varpi^{-b_{\ell-1}} u_\ell v_{\ell-1}}{d} & & & \\
& & & \frac{\varpi^{-b_\ell}\left(w - \sum_{i=\ell+1}^{n-1} u_i v_i\right)}{d} & -\frac{\varpi^{-b_\ell} u_{\ell+1}}{u_\ell} & \cdots & -\frac{\varpi^{-b_\ell} u_{n-1}}{u_\ell} & \frac{\varpi^{-b_\ell}}{u_\ell} \\
& & & \frac{\varpi^{-b_{\ell+1}} u_\ell v_{\ell+1}}{d} & \varpi^{-b_{\ell+1}} & & \\
& & & \vdots & & \ddots & \\
& & & \frac{\varpi^{-b_{n-1}} u_\ell v_{n-1}}{d} & & & \varpi^{-b_{n-1}} \\
& & & -\frac{u_\ell}{d} & & & & 0
\end{pmatrix}.
$$

Proposition 8.12. $\ell \neq 1$.

Proof. Suppose $\ell = 1$. Then

$$
g E_{nn} g^{-1} = \begin{pmatrix}
\frac{1}{d} & 0 & \cdots & \cdots & 0 \\
-\frac{u_1 v_2}{d} & 0 & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots \\
-\frac{u_1 v_{n-1}}{d} & 0 & \cdots & \cdots & 0 \\
-\frac{w u_1}{d} & 0 & \cdots & \cdots & 0
\end{pmatrix}.
$$

Let $k = (1 + d\varpi)^{-1}(I_n + d\varpi E_{nn}) \in K_1(\mathfrak{p}^m)$. Then

$$
k' = g k g^{-1} = (1 + d\varpi)^{-1}(I_n + d\varpi g E_{nn} g^{-1}) \in K'.
$$

Because g is relevant, we have

$$
1 = \chi(k') = \psi(-t_n w u_1 (1 + d\varpi)^{-1}),
$$

a contradiction. ∎

Thus $\ell \geq 2$.

Proposition 8.13. *We have:*

(a) $b_1 > b_2 > \cdots > b_{\ell-1}$ *if* $\ell > 2$.

(b) *Without loss of generality, we can assume that* $v_1 = \cdots = v_{\ell-1} = 0$.

Proof.   We are going to prove the proposition by backward induction. First of all, it is not hard to show that

$$gE_{\ell-1,\ell-1}g^{-1} = E_{\ell-1,\ell-1} + \frac{u_\ell v_{\ell-1}}{d}E_{\ell-1,\ell}.$$

If $d \nmid v_{\ell-1}$, then $v_{\ell-1} \neq 0$ and $\frac{d}{v_{\ell-1}} \in \mathfrak{p}$. Let

$$k = I_n + \frac{d}{v_{\ell-1}}E_{\ell-1,\ell-1} \in K_1(\mathfrak{p}^m).$$

Then

$$k' = gkg^{-1} = I_n + \frac{d}{v_{\ell-1}}gE_{\ell-1,\ell-1}g^{-1} = I_n + \frac{d}{v_{\ell-1}}E_{\ell-1,\ell-1} + u_\ell E_{\ell-1,\ell} \in K'.$$

Because g satisfies (8.1), $1 = \chi(k') = \psi(t_{\ell-1}u_\ell)$, a contradiction. Therefore $d|v_{\ell-1}$. Now adding the $\frac{v_{\ell-1}u_\ell}{d}$-multiple of the ℓ-th row of g to the $(\ell-1)$-st row, we can replace g by $(I_n + \frac{v_{\ell-1}u_\ell}{d}E_{\ell-1,\ell})g \in K'g$, and therefore we can assume that $v_{\ell-1} = 0$.

Suppose now for some $1 < \ell' \leq \ell - 1$ that

(8.6) $b_{\ell'} > b_{\ell'+1} > \cdots > b_{\ell-1}$ and $v_{\ell'} = \cdots = v_{\ell-1} = 0.$

We will show that the above holds as well for $\ell'-1$ in place of ℓ'. Suppose to the contrary that $b_{\ell'-1} \leq b_{\ell'}$. Then since

$$gE_{\ell'-1,\ell'}g^{-1} = \varpi^{b_{\ell'-1}-b_{\ell'}}E_{\ell'-1,\ell'},$$

we see that

$$k = I_n + \varpi^{b_{\ell'}-b_{\ell'-1}}E_{\ell'-1,\ell'} \in K_1(\mathfrak{p}^m).$$

Then

$$k' = gkg^{-1} = I_n + \varpi^{b_{\ell'}-b_{\ell'-1}}gE_{\ell'-1,\ell'}g^{-1} = I_n + E_{\ell'-1,\ell'} \in K'.$$

Because g satisfies (8.1), $1 = \chi(k') = \psi(t_{\ell'-1})$, a contradiction. This proves the first part of (8.6) for $\ell'-1$.

Next, we note that

$$gE_{\ell-1,\ell'-1}g^{-1} = \varpi^{b_{\ell-1}-b_{\ell'-1}}\left(E_{\ell-1,\ell'-1} + \frac{u_\ell v_{\ell'-1}}{d}E_{\ell-1,\ell}\right).$$

If $d \nmid v_{\ell'-1}$, then $v_{\ell'-1} \neq 0$ and $\frac{d}{v_{\ell'-1}} \in \mathfrak{p}$. By the above, $b_{\ell'-1} > b_{\ell'} > \cdots > b_{\ell-1}$. Therefore

$$k = I_n + \varpi^{b_{\ell'-1}-b_{\ell-1}}\frac{d}{v_{\ell'-1}}E_{\ell-1,\ell'-1} \in K_1(\mathfrak{p}^m),$$

and

$$k' = gkg^{-1} = I_n + \varpi^{b_{\ell'-1}-b_{\ell-1}}\frac{d}{v_{\ell'-1}}gE_{\ell-1,\ell'-1}g^{-1}$$

$$= I_n + \frac{d}{v_{\ell'-1}}E_{\ell-1,\ell'-1} + u_\ell E_{\ell-1,\ell} \in K'.$$

Because g satisfies (8.1), $1 = \chi(k') = \psi(t_{\ell-1}u_\ell)$, a contradiction. Therefore $d|v_{\ell'-1}$. Adding the $\frac{v_{\ell'-1}u_\ell}{d}$-multiple of the ℓ-th row of g to the $(\ell'-1)$-st row, we can replace g by $(I_n + \frac{v_{\ell'-1}u_\ell}{d}E_{\ell'-1,\ell})g \in K'g$, allowing us to assume that $v_{\ell'-1} = 0$. This proves the second half of (8.6) for $i = \ell' - 1$ and completes the induction. ∎

From now on, we assume $v_1 = v_2 = \cdots = v_{\ell-1} = 0$.

Proposition 8.14. *Suppose $\ell \le i \le n - 1$. If $u_i \ne 0$, then $b_1 + 1 < b_i$.*

Proof. If $i = \ell$, then

$$gE_{i1}g^{-1} = \varpi^{b_i - b_1}u_i E_{n1}.$$

Likewise, for $\ell < i \le n - 1$, we have

$$gE_{i1}g^{-1} = \varpi^{b_i - b_1}(E_{i1} + u_i E_{n1}).$$

In either case, suppose $b_1 + 1 \ge b_i$. Then $k = I_n + \varpi^{b_1 - b_i + 1}E_{i1} \in K_1(\mathfrak{p}^m)$ since $i < n$. Furthermore,

$$k' = gkg^{-1} = I_n + \varpi^{b_1 - b_i + 1}gE_{i1}g^{-1} = I_n + \delta_{i,\ell}\varpi E_{i1} + \varpi u_i E_{n1} \in K'$$

for the Kronecker function $\delta_{i,\ell}$. Because g satisfies (8.1), the above implies $1 = \chi(k') = \psi(t_n u_i)$, a contradiction since $u_i \ne 0$ is a unit. ∎

Proposition 8.15. *Suppose $\ell + 1 \le i \le n - 1$. If $u_i \ne 0$, then $v_i \in \mathfrak{p}$.*

Proof. Note that

$$gE_{\ell-1,i}g^{-1} = \varpi^{b_{\ell-1}-b_i}\left(\tfrac{u_\ell v_i}{d}E_{\ell-1,\ell} + E_{\ell-1,i}\right).$$

By Proposition 8.13 (a) and the above proposition, $b_i > b_1 > b_{\ell-1}$. Therefore

$$k = I_n + d\varpi^{b_i - b_{\ell-1}}E_{\ell-1,i} \in K_1(\mathfrak{p}^m)$$

and

$$k' = gkg^{-1} = I_n + d\varpi^{b_i - b_{\ell-1}}gE_{\ell-1,i}g^{-1} = I_n + u_\ell v_i E_{\ell-1,\ell} + dE_{\ell-1,i} \in K'.$$

Because g satisfies (8.1), $1 = \chi(k') = \psi(t_{\ell-1}u_\ell v_i)$. This means $v_i \in \mathfrak{p}$ since $t_\ell u_\ell \in \mathfrak{o}^*$. ∎

Proposition 8.16. $\left(w - \sum_{i=\ell+1}^{n-1} u_i v_i\right) \in \mathfrak{p}$.

Proof. We have

$$gE_{\ell-1,\ell}g^{-1} = \varpi^{b_{\ell-1}-b_\ell}\left(\frac{w - \sum_{i=\ell+1}^{n-1} u_i v_i}{d}E_{\ell-1,\ell} - \sum_{i=\ell+1}^{n-1}\frac{u_i}{u_\ell}E_{\ell-1,i} + \frac{1}{u_\ell}E_{\ell-1,n}\right).$$

By Propositions 8.13 (a) and 8.14, $b_\ell > b_1 \geq b_{\ell-1}$. Thus

$$k = I_n + d\varpi^{b_\ell-b_{\ell-1}}E_{\ell-1,\ell} \in K_1(\mathfrak{p}^m)$$

and

$$k' = gkg^{-1} = I_n + d\varpi^{b_\ell-b_{\ell-1}}gE_{\ell-1,\ell}g^{-1} \in K'.$$

Since g is relevant, $1 = \chi(k') = \psi(t_{\ell-1}(w - \sum_{i=\ell+1}^{n-1} u_i v_i))$. The conclusion follows. ∎

Proposition 8.17. *If $[g]$ is not divisible by g_χ, then $[g]$ is relevant only if it contains an element of the form* (8.3) *with $u_1 = u_2 = \cdots = u_{n-1} = 0$.*

Proof. If not all u_i equal 0, let ℓ be the smallest index such that $u_\ell \neq 0$. By Proposition 8.12, $\ell \geq 2$. Then by Proposition 8.15, either $u_i = 0$ or $\varpi|v_i$ for $i = \ell+1, \ldots, n-1$. Therefore $\varpi|u_i v_i$ for $i = \ell+1, \ldots, n-1$. By Proposition 8.16, it follows that $\varpi|w$, contradicting the fact that w is a unit. ∎

8.4.2. Case (2): $u_1 = u_2 = \cdots = u_{n-1} = 0$

Here we assume that g is not divisible by g_χ, and that it is given by (8.3) with all u_i equal to 0. Because w is a unit, we can replace g by the diagonal matrix

$$(8.7) \quad \begin{pmatrix} w^{-1} & & \\ & \ddots & \\ & & w^{-1} \end{pmatrix}\begin{pmatrix} 1 & & -\frac{v_1}{w} \\ & \ddots & \vdots \\ & 1 & \frac{-v_{n-1}}{w} \\ & & 1 \end{pmatrix}\begin{pmatrix} \varpi^{b_1} & & v_1 \\ & \ddots & \vdots \\ & \varpi^{b_{n-1}} & v_{n-1} \\ & & \frac{w}{} \end{pmatrix}\begin{pmatrix} w & & \\ & \ddots & \\ & & w \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \varpi^{b_1} & & \\ & \ddots & \\ & & \varpi^{b_{n-1}} \\ & & 1 \end{pmatrix} \in HgK_1(\mathfrak{p}^m) = [g].$$

Suppose that g is relevant. Then for $r_1, \ldots, r_n \in \mathfrak{o}$, the condition

$$g\begin{pmatrix} 1 & r_1 & & \\ & 1 & r_2 & \\ & & \ddots & \ddots & \\ & & & & r_{n-1} \\ \varpi^m r_n & & & & 1 \end{pmatrix}g^{-1} = \begin{pmatrix} 1 & r_1\varpi^{b_1-b_2} & & \\ & 1 & r_2\varpi^{b_2-b_3} & \\ & & \ddots & \ddots & \\ & & & 1 & r_{n-1}\varpi^{b_{n-1}} \\ \varpi^{m-b_1}r_n & & & & 1 \end{pmatrix} \in K',$$

implies that the above matrix belongs to $\ker\chi$, i.e.

$$(8.8) \quad \psi(t_1\varpi^{b_1-b_2}r_1 + \cdots + t_{n-1}\varpi^{b_{n-1}}r_{n-1} + t_n\varpi^{m-1-b_1}r_n) = 1.$$

Set $r_1 = \cdots = r_{n-1} = 0$. Then (8.8) gives $\psi(t_n \varpi^{m-1-b_1} r_n) = 1$ for all $r_n \in \varpi^{b_1-m+1} \mathfrak{o} \cap \mathfrak{o}$. If $b_1 \geq m - 1$, then we can take $r_n = \varpi^{b_1-m+1}$ to get $\psi(t_n) = 1$, a contradiction. Hence $b_1 < m - 1$. Similarly, for any index $1 \leq j < n$, set $r_i = 0$ for all $i \neq j$. Then for all $r_j \in \varpi^{b_{j+1}-b_j} \mathfrak{o} \cap \mathfrak{o}$, (8.8) gives $\psi(t_j \varpi^{b_j - b_{j+1}} r_j) = 1$. If $b_{j+1} \geq b_j$, then we can take $r_j = \varpi^{b_{j+1}-b_j}$ and arrive at a contradiction. We conclude that

$$(8.9) \qquad 0 = b_n < b_{n-1} < b_{n-2} < \cdots < b_1 < m - 1.$$

Conversely, it is clear from the above that if (8.9) holds, then (8.8) holds, so g is relevant. This proves the following.

Proposition 8.18. *Suppose g is the diagonal matrix* (8.7). *Then g is relevant if and only if* (8.9) *holds.*

Proof of Theorem 8.10. By Proposition 8.4, $[g] = [g_\chi^a h] = g_\chi^a[h]$, where $[h]$ is not divisible by g_χ and $1 \leq a \leq n-1$. By Proposition 8.5, $[g]$ is relevant if and only if $[h]$ is relevant. By Proposition 8.17 and the above discussion, $[h]$ is relevant if and only if it contains a matrix of the form (8.5). ∎

REFERENCES

[AL]   M. Adrian and B. Liu, *The local Langlands correspondence for simple supercuspidal representations of* $\mathrm{GL}_n(F)$, arXiv:1310.2585.

[BH1] C. Bushnell and G. Henniart, *The Local Langlands Correspondence for* $\mathrm{GL}(2)$, Springer, 2006.

[BH2] ——, Langlands parameters for epipelagic representations of GL_n, *Math. Ann.*, **358(1-2)** (2014), 433-463.

[BK]   C. Bushnell and P. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Annals of Mathematics Studies, 129. Princeton University Press, Princeton, NJ, 1993.

[C1]   H. Carayol, Représentations supercuspidales de GL_n, *C. R. Acad. Sci. Paris Sr. A-B*, **288(1)** (1979), A17-A20.

[C2]   ——, Représentations cuspidales du groupe linéaire, *Ann. Sci. École Norm. Sup. (4)*, **17(2)** (1984), 191-225.

[GK] I. M. Gelfand and D. A. Kajdan, *Representations of the group GL(n,K) where K is a local field*, Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), pp. 95-118. Halsted, New York, 1975.

[G] B. Gross, Irreducible cuspidal representations with prescribed local behavior, *Amer. J. Math.*, **133(5)** (2011), 1231-1258.

[GF] B. Gross and E. Frenkel, A rigid irregular connection on the projective line, *Ann. of Math. (2)*, **170(3)** (2009), 1469-1512.

[GR] B. Gross and M. Reeder, Arithmetic invariants of discrete Langlands parameters, *Duke Math. J.*, **154(3)** (2010), 431-508.

[He] G. Henniart, Sur l'unicité des types pour GL$_2$, *Appendix to Duke Math. J.*, **115(2)** (2002), 205-310.

[Ho] R. Howe, *Hecke algebras and p-adic* GL$_n$, Representation theory and analysis on homogeneous spaces (New Brunswick, NJ, 1993), 65-100, Contemp. Math., **177**, Amer. Math. Soc., Providence, RI, 1994.

[HNY] J. Heinloth, B. Ngô and Z. Yun, Kloosterman sheaves for reductive groups, *Ann. of Math. (2)*, **177(1)** (2013), 241-310.

[J] H. Jacquet, A correction to "Conducteur des représentations du groupe linéaire", *Pac. J. Math.*, **260(2)** (2012), 515-525.

[JPSS] H. Jacquet, I. Pitateskii-Shapiro and J. Shalika, Conducteur des représentations du groupe linéaire, *Math. Ann.*, **256(2)** (1981), 199-214.

[KL1] A. Knightly and C. Li, *Traces of Hecke operators*, Mathematical Surveys and Monographs, 133. Amer. Math. Soc., Providence, RI, 2006.

[KL2] ——, Modular *L*-values of cubic level, *Pac. J. Math.*, **260(2)** (2012), 527-563.

[S] R. Schmidt, Some remarks on local newforms for GL(2), *J. Ramanujan Math. Soc.*, **17(2)** (2002), 115-147.

Andrew Knightly
Department of Mathematics & Statistics
University of Maine
Neville Hall, Orono, ME 04469-5752
USA
E-mail: knightly@math.umaine.edu

Charles Li
Department of Mathematics
The Chinese University of Hong Kong
Shatin
Hong Kong
E-mail: charles@charlesli.org