MAT 563 HOMEWORK 3: DUE FRIDAY, FEB. 5

(1) Show that $A_n$ is generated by the 3-cycles, i.e. every element of $A_n$ is a product of 3-cycles. (Hint: reduce to the case where $\sigma = \tau_1 \tau_2$ is a product of two 2-cycles.)

(2) Let $G = \langle a \rangle$ be a cyclic group of order $n$. Use the first isomorphism theorem to prove that $G \cong \mathbb{Z}/n\mathbb{Z}$.

(3) Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, c \in \mathbb{R}, ad \neq 0 \right\}$.
   
   (a) Show that $B$ is a subgroup of $\text{GL}_2(\mathbb{R})$, and that $B$ is nonabelian.
   
   (b) Let $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$, called the unipotent subgroup of $\text{GL}_2(\mathbb{R})$. Prove that $B/N \cong \mathbb{R}^* \times \mathbb{R}^*$, where $\mathbb{R}^* \times \mathbb{R}^*$ is the set $\left\{ (x, y) \mid x, y \in \mathbb{R}^* \right\}$ with the abelian group operation $(x, y) \cdot (a, b) = (xa, yb)$. (Use the 1st isomorphism theorem.)
   
   (c) Show that $N$ is the commutator subgroup of $B$. (By part (b) you know that $N$ contains it.)

(4) Show that $V = \{ e, (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3) \}$ is a normal subgroup of $S_4$. The quotient $S_4/V$ is isomorphic to a familiar group. Can you find it?

(5) Consider a tetrahedron $T$ (a pyramid with 4 congruent triangular sides). Label the vertices 1, 2, 3, 4. Find the symmetry group of $T$, i.e. the subgroup of $S_4$ consisting of permutations which give rotations of $T$. 

**Extra problems, not to be handed in:**

(6) Let $G$ be a group with $|G| = pq$, where $p$ and $q$ are primes. Show that every proper subgroup of $G$ is cyclic.

(7) Let $H = \langle (1 \ 2 \ 4) \rangle < A_4$. Determine all right and left cosets of $H$ in $A_4$. Based on your computation, is $H \triangleleft A_4$?

(8) If every element of a nontrivial group $G$ has even order, show that there exists $a \neq e$ such that $a^2 = e$.

(9) (This was assigned last semester) In $S_n$, an **elementary 2-cycle** is a 2-cycle of the form $(a \ a+1)$.

   (a) Prove that every 2-cycle is the product of an odd number of elementary 2-cycles.
   
   **Hint:** Let $(a \ b)$ be the 2-cycle, with $b > a$. Use induction on $b - a$. Base your idea on the example: $(2 \ 5) = (2 \ 3)(3 \ 5)(2 \ 3) = (2 \ 3)(3 \ 4)(4 \ 5)(3 \ 4)(2 \ 3)$. 

   \[ (2 \ 5) = (2 \ 3)(3 \ 5)(2 \ 3) = (2 \ 3)(3 \ 4)(4 \ 5)(3 \ 4)(2 \ 3) \]
(b) Conclude that \( S_n \) is generated by the elementary 2-cycles.

(c) Show that \( S_n \) is also generated by the two elements \((1\ 2)\) and \((1\ 2\ \ldots\ n)\).

(10) (This was assigned last semester) In a group \( G \), the **conjugacy class** of an element \( a \in G \) is the set
\[
K(a) = \{ gag^{-1} \mid g \in G \}.
\]
A **conjugate** of \( a \) is an element of \( K(a) \), i.e. any element of the form \( gag^{-1} \). In this exercise you will investigate conjugacy classes in \( S_n \).

(a) Show that a conjugate of a \( k \)-cycle is again a \( k \)-cycle: Choose any \( k \)-cycle \( \phi \in S_n \). Then
\[
\phi = (a_1\ a_2\ \ldots\ a_k),
\]
for some distinct \( a_i \in \{1, 2, \ldots, n\} \). Let \( \tau \in S_n \) be any permutation. Compute \( \tau \phi \tau^{-1} \). (Hint: where does \( \tau \phi \tau^{-1} \) send \( \tau(a_1) \)?)

(b) Let \( \phi \) be a \( k \)-cycle as above. Show that
\[
K(\phi) = \{ k\text{-cycles} \in S_n \}.
\]
(You showed \( \subset \) in part (a). You just have to show \( \supset \).)