(1) Let $G$ be a group of order 15. Show that $G$ contains an element of order 3.

(2) Let $G$ be a group, and let $N < G$ be a subgroup of index 2. Prove that $N$ is normal in $G$.

(3) Let $Z$ be the center of a group $G$ (recall $Z < G$). Prove that if the group $G/Z$ is cyclic, then $G$ is abelian.

(4) Let $G$ be a group and let $G'$ be the subgroup generated by the set
\[ \{aba^{-1}b^{-1} | a, b \in G \}. \]
This is the commutator subgroup of $G$. It measures the degree to which $G$ fails to be abelian. (E.g. $G' = \{e\}$ if $G$ is abelian.)

(a) Prove that $G' < G$ and that $G/G'$ is abelian.

(b) If $H < G$ and $G/H$ is abelian, prove that $G' < H$. (Thus $G/G'$ is the maximal abelian quotient of $G$.)

(5) Determine all homomorphisms from $Z$ to $Z$. 