MATH 261

MIDTERM SOLUTIONS

October 24, 2017

1. (10 pts) Suppose P, Q, R are statements, with:

$$P \text{ False}, \quad Q \text{ False}, \quad R \text{ True}.$$

Suppose also that A, B are arbitrary sets. Label the following statements as True or False. No justification or work is needed.

$(\sim P) \wedge R$	*True*
$\sim (P \wedge R)$	*True*
$Q \implies \sim (P \wedge \sim R)$ (False implies anything)	*True*
$(A \cap B)^c = A^c \cap B^c$ (DeMorgan: $(A \cap B)^c = A^c \cup B^c$.)	*False*
A, B are disjoint $\implies A \cap B = \{\emptyset\}$ ($\{\emptyset\}$ is not an empty set!)	*False*

2. (10 pts) Prove that the sum of a rational number with an irrational number is irrational.

Let x, y be real numbers with x rational and y irrational. Suppose to the contrary that $x+y$ is rational. Then we may write $x = \frac{a}{b}$ and $x + y = \frac{c}{d}$ for some integers a, b, c, d. Then

$$y = (x+y) - x = \frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{bd} \in \mathbf{Q},$$

which contradicts the irrationality of y. Therefore $x + y$ is irrational.

3. (14 pts) Let X be a set.

(a) Define the power set $\mathcal{P}(X)$.

The power set $\mathcal{P}(X)$ is the set of all subsets of X.

(b) Prove that there exists a unique element $B \in \mathcal{P}(X)$ such that $A \cap B = A$ for all $A \in \mathcal{P}(X)$.

Let $B = X \in \mathcal{P}(X)$. Then for any $A \in \mathcal{P}(X)$, $A \cap X = A$ since $A \subseteq X$. This proves the existence of such a set.

For the uniqueness, suppose $B \in \mathcal{P}(X)$ has the property that $A \cap B = A$ for all $A \subseteq X$. Then taking $A = X$ we have

$$X = X \cap B = B.$$

So $B = X$, which shows that B is unique.

4. (16 pts) Prove that for all integers $n \geq 0$,

$$\sum_{j=0}^{n} 2^j = 2^{n+1} - 1.$$

We induct on n. For the base case, when $n = 0$, we have $2^0 = 1 = 2 - 1$, as needed. Now suppose for some $k \geq 0$ that

$$\sum_{j=0}^{k} 2^j = 2^{k+1} - 1.$$

Then

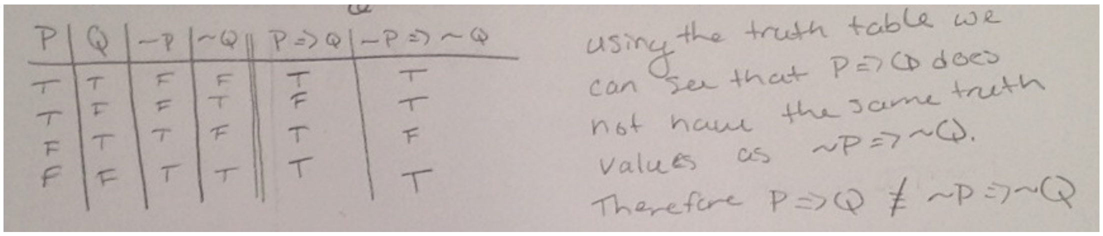
$$\sum_{j=0}^{k+1} 2^j = 2^{k+1} + \sum_{j=0}^{k} 2^j = 2^{k+1} + (2^{k+1} - 1) = 2 \cdot 2^{k+1} - 1$$
$$= 2^{(k+1)+1} - 1,$$

which establishes the desired formula when $n = k + 1$. By induction, the formula holds for all $n \geq 0$.

5. (12 pts) Let P and Q be statements.

(a) Use a truth table to show that $P \implies Q$ is NOT equivalent to $\sim P \implies \sim Q$. (Your table should have 6 columns, including $\sim P$ and $\sim Q$.)

From a student exam:



P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$\sim P \Rightarrow \sim Q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

using the truth table we can see that $P \Rightarrow Q$ does not have the same truth values as $\sim P \Rightarrow \sim Q$.
Therefore $P \Rightarrow Q \not\equiv \sim P \Rightarrow \sim Q$

(b) Give an implication (without any $\sim$ symbols) that IS equivalent to $\sim P \implies \sim Q$. (No explanation needed.)

$Q \implies P$ is the contrapositive of the given implication. We know that any implication is equivalent to its contrapositive.

6. (14 pts) Consider the following statement:
There exists a real number y such that $xy = y$ for all real numbers x.

(a) Write the statement in symbolic notation (no words allowed).

$$\exists y \in \mathbf{R}, \forall x \in \mathbf{R}, xy = y.$$

(b) Write the negation of the above statement using symbolic notation. Your final answer must NOT include the negation symbol $\sim$.

$$\forall y \in \mathbf{R}, \exists x \in \mathbf{R}, xy \neq y.$$

(c) Write (in words) the negation of the original statement.

There is a real number y such that for some real number x, $xy \neq y$.

(d) Which is true: the original statement, or its negation? (Why?)

The original statement is true, since we may take $y = 0$: $x \cdot 0 = 0$ for all x.

7. (12 pts) Consider the following statement about an integer x:

If $x^2 + 2x - 5$ *is odd, then* x *is even.*

(a) State the hypothesis you would adopt if attempting a direct proof of the statement. What conclusion would you seek?

Hypothesis assumed: $x^2 + 2x - 5$ is odd.

Conclusion sought: x is even.

(b) State the hypothesis you would adopt if attempting a contrapositive proof of the statement. What conclusion would you seek?

Hypothesis assumed: x is odd.

Conclusion sought: $x^2 + 2x - 5$ is even.

(c) Which of the two methods is better in this case? Why?

Contrapositive is preferable here, since it is straightforward using substitution to infer information about $x^2 + 2x - 5$ from information about x, whereas it is not so clear how to obtain information about x from a hypothesis involving $x^2 + 2x - 5$.

(d) Prove the statement using the best method.

Suppose x is odd. Then $x = 2k + 1$ for some $k \in \mathbf{Z}$. So

$$x^2+2x-5 = (2k+1)^2+2(2k+1)-5 = 4k^2+4k+1+4k+2-5 = 4k^2+8k-2 = 2(2k^2+4k-1),$$

which is even. Therefore if $x^2 + 2x - 5$ is odd, x must be even.

8. (12 pts) Prove that if p and q are distinct prime numbers, then

$$pq\mathbf{Z} = p\mathbf{Z} \cap q\mathbf{Z}.$$

For the $\supseteq$ containment, use Euclid's Lemma.

Any element $x \in pq\mathbf{Z}$ is a multiple of p and a multiple of q, which means that $x \in p\mathbf{Z}$ and $x \in q\mathbf{Z}$, i.e., $x \in p\mathbf{Z} \cap q\mathbf{Z}$. Therefore $pq\mathbf{Z} \subseteq p\mathbf{Z} \cap q\mathbf{Z}$.

On the other hand, if $x \in p\mathbf{Z} \cap q\mathbf{Z}$, then we may write

$$x = pa = qb$$

for some integers a, b. In particular, $p|qb$. Since p is prime, Euclid's lemma implies that either $p|q$ or $p|b$. Because q is also prime and not equal to p, we know that $p \nmid q$. Therefore it must be the case that $p|b$. Writing $b = pk$ for some $k \in \mathbf{Z}$, we have

$$x = qb = qpk = pqk \in pq\mathbf{Z}.$$

Hence $p\mathbf{Z} \cap q\mathbf{Z} \subseteq pq\mathbf{Z}$, which, in view of the reverse containment shown above, proves that $pq\mathbf{Z} = p\mathbf{Z} \cap q\mathbf{Z}$.