MAT 261 MIDTERM 1 REVIEW PROBLEM SOLUTIONS

MIDTERM 1 IS IN CLASS FRI, OCTOBER 20.

(1) Suppose P is true, Q is true, R is false, and S is false. Give the truth values of each of the following.
 (a) $P \vee R$: T
 (b) $\sim P \vee R \vee S$: F
 (c) $P \vee Q \implies S$: F
 (d) $\sim (P \vee S)$: F
 (e) $P \wedge S \implies R$: T

(2) Write the following statements in words. The only symbols allowed are x and y.
 (a) $(\forall x)(\exists y)(y > x) \wedge (y < 2x)$.
 For all x, there exists y such that y is greater than x and less than twice x.
 (b) $(\exists x)(\forall y)(\sim (x + y \in \mathbf{Q}))$.
 There exists x such that for all y, the sum of x and y is irrational.

(3) Write the following sentence in symbolic logic.
There is exactly one natural number with the property that its product with some irrational number is rational.

$$(\exists! n \in \mathbf{N})(\exists y \in \mathbf{R} - \mathbf{Q})(ny \in \mathbf{Q}).$$

(4) Give the negation of an implication $P \implies Q$.

$$\sim (P \implies Q) \equiv \sim (\sim P \vee Q) \equiv P \wedge \sim Q.$$

(5) Negate the following statement:
For all $x > 0$, there exists a number y such that if $x < 5$, then $y < x$, and if $x \geq 5$, then $y \geq x$.
 It may be helpful to write the given statement in symbolic logic first:

$$(\forall x > 0)(\exists y)(x < 5 \implies y < x) \wedge (x \geq 5 \implies y \geq x).$$

From the previous problem, we know how to negate an implication. So the negation, in symbolic logic, is

$$(\exists x > 0)(\forall y)((x < 5) \wedge (y \geq x)) \vee ((x \geq 5) \wedge (y < x)).$$

In words: *There exists $x > 0$ such that for all numbers y, either $x < 5$ and $y \geq x$, or $x \geq 5$ and $y < x$.*

(6) Negate the following statement:
For some $x > 0$, there exists $y > 0$ such that $y^2 < x$.
 Again we may choose to start by rewriting the statement in symbolic logic:

$$(\exists x > 0)(\exists y > 0)(y^2 < x).$$

The negation is

$$(\forall x > 0)(\forall y > 0)(y^2 \geq x),$$

or in words:
For all $x > 0$ and all $y > 0$, $y^2 \geq x$.

(7) Suppose A, B, C are sets, with $B \subseteq C$. Prove that $A \cap B \subseteq C$.

Proof. Let $x \in A \cap B$. Then in particular $x \in B$. Since $B \subseteq C$, it follows that $x \in C$, as needed. $\square$

(8) Suppose $C \neq \emptyset$ and $A \times C \subseteq B \times C$. Prove that $A \subseteq B$.

Proof. Let $x \in A$. Since $C \neq \emptyset$, there exists $y \in C$. Then $(x, y) \in A \times C \subseteq B \times C$. This means that $(x, y) \in B \times C$, from which it follows that $x \in B$. This proves that $A \subseteq B$. $\square$

(9) Prove or give a counterexample: Suppose A and B are disjoint and $B \subseteq C$. Then A and C are disjoint.

This is false. Consider $A = \{1\}$, $B = \emptyset$, and $C = \{1\}$. Then $A \cap B = \emptyset$, so A and B are disjoint, and $B \subseteq C$ since the empty set is a subset of every set. However A and C are certainly not disjoint since they have an element in common (in fact are equal).

(10) Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof. Suppose $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and $y \in B \cup C$. So either $y \in B$ or $y \in C$. If $y \in B$, then $(x, y) \in A \times B$. If $y \in C$, then $(x, y) \in A \times C$. This shows that either $(x, y) \in A \times B$ or $(x, y) \in A \times C$, i.e. $(x, y) \in (A \times B) \cup (A \times C$. Hence $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

Conversely, suppose $(x, y) \in (A \times B) \cup (A \times C)$. So either $(x, y) \in A \times B$, or $(x, y) \in A \times C$. This means that either $y \in B$ or $y \in C$, i.e. $y \in B \cup C$. Hence $(x, y) \in A \times (B \cup C)$. This shows that $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Since the two given sets are subsets of each other, they are equal. $\square$

(11) Find the power set of $A = \{\emptyset, \{1, \{2\}\}, 3\}$.

A systematic way to find all subsets is to list in order all subsets of a given size. We find that

$$\mathcal{P}(A) = \Big\{\emptyset, \{\emptyset\}, \{\{1, \{2\}\}\}, \{3\}, \{\emptyset, \{1, \{2\}\}\}, \{\emptyset, 3\}, \{\{1, \{2\}\}, 3\}, \{\emptyset, \{1, \{2\}\}, 3\}\Big\}.$$

Don't forget that if a set has n elements, then its power set has 2^n elements. (Can you prove it by induction?)

(12) Prove that x is even if and only if $10x^2 + 5x - 2$ is even.

Proof. Suppose x is even. Then $x = 2k$ for some $k \in \mathbf{Z}$, and

$$10x^2 + 5x - 2 = 10(2k)^2 + 5(2k) - 2 = 2(20k^2 + 5k - 1)$$

is also even. Conversely, if x is odd, then $x = 2k+1$ for some $k \in \mathbf{Z}$. Hence

$$10x^2 + 5x - 2 = 10(2k + 1)^2 + 5(2k + 1) - 2$$
$$= 10(4k^2 + 4k + 1) + (10k + 5) - 2 = 10(4k^2 + 4k + 1) + (10k + 5 - 3) + 1$$
$$= 2(5(4k^2 + 4k + 1) + (5k - 1)) + 1$$

is odd. This proves the result. $\square$

(13) Let x be an irrational real number. Prove that $1/x$ is also irrational. *We remark that $x \neq 0$ since 0 is rational. So $1/x$ makes sense as a real number.*

Proof. Suppose to the contrary that $1/x$ is rational. Then we may write

$$\frac{1}{x} = \frac{a}{b}$$

where a, b are integers, with $b \neq 0$. In fact, $a \neq 0$ as well, because $1/x \neq 0$. Taking reciprocals, $x = \frac{b}{a}$, showing that x is rational, a contradiciton. Hence $1/x$ is irrational. □

(14) Prove that for every $x \in \mathbf{R}$ there exists a unique $y \in \mathbf{R}$ such that

$$(x+1)^2 - x^2 = 2y - 1.$$

Solution: For the existence of y, given $x \in \mathbf{R}$, let $y = \frac{(x+1)^2 - x^2 + 1}{2}$. Then

$$2y - 1 = (x+1)^2 - x^2,$$

as needed. For uniqueness, suppose

$$2y - 1 = (x+1)^2 - x^2 = 2z - 1$$

for some $y, z \in \mathbf{R}$. Then

$$2y - 1 = 2z - 1.$$

Adding 1 to both sides and then dividing by 2, we see that $y = z$, as needed.

(15) Let $n \geq 2$, and let $A_1, \ldots, A_n$ be sets in some universe. Using induction, prove the following generalization of DeMorgan's Law:

$$\left(\bigcap_{i=1}^{n} A_i \right)^c = \bigcup_{i=1}^{n} A_i^c.$$

Proof. We induct on n. For the base case, suppose $n = 2$. By DeMorgan's Law, we know that

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c,$$

as needed. For the inductive step, suppose for some $k \geq 2$ that for any collection of k sets $A_1, \ldots A_k$,

$$\left(\bigcap_{i=1}^{k} A_i \right)^c = \bigcup_{i=1}^{k} A_i^c.$$

Now suppose we are given $k+1$ sets $A_1, \ldots, A_{k+1}$. Then

$$\left(\bigcap_{i=1}^{k+1} A_i \right)^c = \left(\left[\bigcap_{i=1}^{k} A_i \right] \cap A_{k+1} \right)^c = \left[\bigcap_{i=1}^{k} A_i \right]^c \cup A_{k+1}^c$$

by DeMorgan's Law (the base case). Now applying the inductive hypothesis, the above is

$$= \left[\bigcup_{i=1}^{k} A_i^c \right] \cup A_{k+1}^c = \bigcup_{i=1}^{k+1} A_i^c,$$

which proves the desired identity in the case where $n = k+1$. By induction, the identity holds for all $n \geq 2$. □

Every induction proof needs to have:
(a) Clear statement that you are using induction.
(b) Clear statement and proof of the base case.
(c) Clear statement of the inductive hypothesis.
(d) Concluding statement.

(16) Consider the following incorrect theorem:

Theorem 0.1. *Suppose $x + y = 12$. Then $x \neq 3$ and $y \neq 8$.*

(a) What is wrong with the following proof?

Proof. . Suppose to the contrary that $x = 3$ and $y = 8$. Th $x+y = 11$. This contradicts the hypothesis that $x + y = 12$. Hence $x \neq 3$ and $y \neq 8$. □

Point out exactly where the mistake occurs.
The mistake occurs in the first sentence. This is an attempt at a proof by contradiction, which should involve assuming the negation of the conclusion. In symbolic logic, the conclusion is $(x \neq 3) \wedge (y \neq 8)$. So a proof by contradiction should start by assuming

$$\sim [(x \neq 3) \wedge (y \neq 8)] \equiv (x = 3) \vee (y = 8),$$

i.e., that $x = 3$ OR $y = 8$. The proof incorrectly uses AND where there should be an OR.
Actually, a second similar mistake occurs in the last sentence. The writer incorrectly writes $x \neq 3$ and $y \neq 8$, when it has only been established that $\sim ((x = 3) \wedge (y = 8))$, i.e. that $x \neq 3$ OR $y \neq 8$).

(b) What method should be used when disproving a theorem?
Counterexample.

(c) Disprove Theorem 0.1.
Consider $x = 3$ and $y = 9$. Then $x+y = 3+9 = 12$, so the hypothesis is satisfied. However, the conclusion is not satisfied (since $x = 3$). This counterexample disproves the theorem.

(17) Let p and q be distinct prime numbers.

(a) Prove that

$$pq\mathbf{Z} = p\mathbf{Z} \cap q\mathbf{Z}.$$

For the $\supseteq$ containment, use Euclid's Lemma.

Proof. Let $x \in qp\mathbf{Z}$. Then for some $k \in \mathbf{Z}$, $x = qpk$, which belongs to both $q\mathbf{Z}$ and $p\mathbf{Z}$. Therefore $qp\mathbf{Z} \subseteq q\mathbf{Z} \cap p\mathbf{Z}$.

Now suppose $x \in q\mathbf{Z} \cap p\mathbf{Z}$. Because $x \in q\mathbf{Z}$, $x = qk$ for some $k \in \mathbf{Z}$. We know that $x \in p\mathbf{Z}$ as well, which means that $p|x$, i.e., $p|qk$. By Euclid's Lemma, either $p|q$ or $p|k$. Since q is a prime different from p, we know that $p \nmid q$. Therefore $p|k$, so $k = pr$ for some $r \in \mathbf{Z}$. Putting this together, we have $x = pqr \in pq\mathbf{Z}$. Hence $p\mathbf{Z} \cap q\mathbf{Z} \subseteq qp\mathbf{Z}$. It now follows that $p\mathbf{Z} \cap q\mathbf{Z} = pq\mathbf{Z}$. □

(b) Does the same equality hold if we remove the hypothesis that p and q are distinct?
No, it does not. If $p = q$, then $p \in p\mathbf{Z}$, but $p \notin p^2\mathbf{Z}$. So $p\mathbf{Z} \cap p\mathbf{Z} = p\mathbf{Z} \neq p^2\mathbf{Z}$.