MAT 261 MIDTERM 1 REVIEW

MIDTERM 1 IS IN CLASS FRI, OCTOBER 20.

Review of topics covered:

1. Sets

Concepts to know: subset, intersection, union, power set, Cartesian product, complement, DeMorgan's Laws.

2. Logic

Know the definition of *statement*. Be able to identify a statement from a non-statement.

Know the meaning of and ($\wedge$), or ($\vee$), and not ($\sim$). They allow us to form *compound statements* from several component statments.

Know the meaning of equivalent statements ($P \equiv Q$). Be able to use truth tables to verify an equivalence. (Examples include logic version of DeMorgan's Laws.)

Implication ($P \implies Q$): Know the meaning and the associated truth table.
Know the meaning of the converse and contrapositive of a given implication.
Know and be able to prove (via truth table) that an implication is equivalent to its contrapositive, but not to its converse.
Know and be able to prove (via truth table) that $(P \implies Q) \equiv (\sim P \vee Q)$.

3. Predicate logic

There are three quantifiers: $\forall$, $\exists$ and $\exists!$. Be sure to know their meanings, and how to translate logical notation into words and vice versa.

Be able to negate a statement involving quantifiers.

4. Miscellaneous definitions

Know what it means for an integer to be even or odd.
Know what it means for a real number to be rational or irrational.

5. Proofs

Proving one set is a subset of another set. To prove $A \subseteq B$, start your proof by writing

Let $x \in A$.

Now use the definitions of A and B to show that $x \in B$. Once you've shown that $x \in B$, write:

Therefore, $A \subseteq B$.

Proving two sets are equal. To prove $A = B$, first show that $A \subseteq B$ (using the above method), and then prove $B \subseteq A$. Once this is done, you may conclude that $A = B$.

Proving an implication. There are two strategies for proving a statement of the form $P \implies Q$.

Direct proof. Assume P, and try to prove Q.

Contrapositive proof. Assume $\sim Q$ and try to prove $\sim P$.

Be able to identify the hypotheses to be adopted in each of the methods, and use this information to choose the best strategy.

Bidirectional proofs. Understand that proving a statement of the form *"P if and only if Q"* involves giving two proofs: The forward direction ($P \implies Q$) AND its converse ($Q \implies P$). (The statement P *if and only if* Q is the same as $P \equiv Q$.) For each direction, choose the best strategy (direct or contrapositive). Of course, you often won't use the same strategy for each direction.

Proof by contradiction. If the statement to be proven involves a "negative-sounding" conclusion, then a proof by contradiction is often the best strategy. Such a proof begins: *Assume, to the contrary, that [statement] is false.* Then try to derive something nonsensical from that premise. Conclude that the statement must therefore be true.

Counterexamples. If your intention is to *disprove* a theorem involving the universal quantifier (i.e. a statement that is allegedly always true) then you should give a counterexample. This is a specific example that satisfies the given hypotheses, but does not satisfy the conclusion.

6. Uniqueness proofs

The statement ($\exists! x$ such that $P(x)$) is really two statements in one: existence and uniqueness. A proof of such a statement requires two separate proofs:

1. To prove existence, produce some element x (found on scratch paper) and show that $P(x)$ is true.
2. To prove uniqueness, suppose for some x and y that $P(x)$ and $P(y)$ are both true. Then prove $x = y$. (You'll note that proving a function is 1-1 is a special instance of a uniqueness proof.)

What not to do. Do not start by assuming that the desired conclusion is true! This may be a good idea on scratch paper as you get a feel for what's going on, but it has no place in a final writeup.

7. The Division Algorithm

Know what it means for one integer a to divide another integer d ($a|d$). When solving a question about divisibility, the Division Algorithm is likely to come to your aid. You should carefully memorize its statement:

Theorem 7.1. *For any integers a, b with $b > 0$, there exist unique integers q and r such that*

i. $a = qb + r$

ii. $0 \leq r < b$.

We proved this existence/uniqueness theorem in class using WOP. You do not need to memorize the existence part of the proof, but you should be able to do the uniqueness. Here it is again:

Proof of uniquness. Suppose, for some $q, q', r, r' \in \mathbf{Z}$ that

$$a = qb + r = q'b + r',$$

and that

$$0 \le r < b, \qquad 0 \le r' < b.$$

We may suppose (wlog) that $r \ge r'$. Then $0 \le r - r' < b$. On the other hand,

$$r - r' = b(q' - q). \tag{1}$$

So $r - r'$ is an integer multiple of b inside the interval $[0, b)$. Hence $r - r' = 0$. It follows from (1) that $q' - q = 0$, so $q = q'$ and $r = r'$. □

Also know: Bezout's Theorem, Euclid's Lemma.

8. Induction and WOP

We have shown that the following two statements are equivalent. We assumed WOP as an axiom and used it to prove the Principle of Induction, but one could do the reverse (as shown on HW6).

Axiom 8.1 (Well Ordering Principle (WOP))**.** *Every nonempty subset of the natural numbers has a least element.*

Theorem 8.1 (Principle of Mathematical Induction (MI))**.** *Suppose for each natural number n that $P(n)$ is a statement. Suppose also that*

1. *$P(1)$ is true, and*
2. *for all $k \ge 1$, $P(k) \implies P(k+1)$.*

Then $P(n)$ is true for all $n \ge 1$.

Strong Induction variant: In place of (2), it is permissible to use the following: (2′) for all $k \ge 1$, $P(1) \wedge P(2) \wedge \cdots \wedge P(k)$ implies $P(k+1)$.

Make sure to practice lots of induction proofs. One of these will surely show up on the midterm.

9. Relations

Not covered on the midterm!

10. Sample Problems

At least one of these will show up verbatim on the actual exam. You may also be asked for definitions of terms mentioned above. Make sure you know them well. No notes or calculators are allowed. Also fair game: HW problems!

(1) Suppose P is true, Q is true, R is false, and S is false. Give the truth values of each of the following.
   (a) $P \vee R$
   (b) $\sim P \vee R \vee S$
   (c) $P \vee Q \implies S$
   (d) $\sim (P \vee S)$
   (e) $P \wedge S \implies R$

(2) Write the following statements in words. The only symbols allowed are x and y.

(a) $(\forall x)(\exists y)(y > x) \wedge (y < 2x)$.
(b) $(\exists x)(\forall y)(\sim (x + y \in \mathbf{Q}))$.

(3) Write the following sentence in logical notation. *There is exactly one natural number with the property that its product with some irrational number is rational.*

(4) Give the negation of an implication $P \implies Q$.

(5) Negate the following statement: *For all $x > 0$, there exists a number y such that if $x < 5$, then $y < x$, and if $x \geq 5$, then $y \geq x$.*

(6) Negate the following statement: *For some $x > 0$, there exists $y > 0$ such that $y^2 < x$.*

(7) Suppose A, B, C are sets, with $B \subseteq C$. Prove that $A \cap B \subseteq C$.

(8) Suppose $C \neq \emptyset$ and $A \times C \subseteq B \times C$. Prove that $A \subseteq B$.

(9) Prove or give a counterexample: Suppose A and B are disjoint and $B \subseteq C$. Then A and C are disjoint.

(10) Prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(11) Find the power set of $A = \{\emptyset, \{1, \{2\}\}, 3\}$.

(12) Prove that x is even if and only if $10x^2 + 5x - 2$ is even.

(13) Let x be an irrational real number. Prove that $1/x$ is also irrational.

(14) Prove that for every $x \in \mathbf{R}$ there exists a unique $y \in \mathbf{R}$ such that

$$(x + 1)^2 - x^2 = 2y - 1.$$

(15) Let $n \geq 2$, and let $A_1, \ldots, A_n$ be sets in some universe. Using induction, prove the following generalization of DeMorgan's Law:

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$

(16) Consider the following incorrect theorem:

Theorem 10.1. *Suppose $x + y = 12$. Then $x \neq 3$ and $y \neq 8$.*

(a) What is wrong with the following proof?

Proof. . Suppose to the contrary that $x = 3$ and $y = 8$. Then $x + y = 11$. This contradicts the hypothesis that $x + y = 12$. Hence $x \neq 3$ and $y \neq 8$. □

Point out exactly where the mistake occurs.

(b) What method should be used when disproving a theorem?
(c) Disprove Theorem 0.1.

(17) Let p and q be distinct prime numbers.
(a) Prove that

$$pq\mathbf{Z} = p\mathbf{Z} \cap q\mathbf{Z}.$$

For the $\supseteq$ containment, use Euclid's Lemma.

(b) Does the same equality hold if we remove the hypothesis that p and q are distinct?