MAT 261 HOMEWORK 6: DUE FRIDAY, OCT. 13

Theorem 0.1 (Well Ordering Principle (WOP)). *Every nonempty subset of the natural numbers has a least element.*

Theorem 0.2 (Principle of Mathematical Induction (PMI)). *Suppose for each natural number n that $P(n)$ is a statement. Suppose also that*

(1) *$P(1)$ is true, and*
(2) *for all $k \geq 1$, $P(k) \implies P(k+1)$.*

Then $P(n)$ is true for all $n \geq 1$.

We proved that in place of (2), one may assume instead:

(2′) for all $k \geq 1$, $\big(P(1) \wedge P(2) \wedge \cdots \wedge P(k)\big) \implies P(k+1)$.

(This is **strong induction**.)

HW8 EXERCISES

(1) Prove that for every $n \geq 1$, the number $4^n + 5$ is a multiple of 3.

(2) In class we assumed WOP as an axiom, and used it to prove PMI. Prove that conversely, PMI implies WOP. Hint: Given a subset $S \subseteq \mathbf{N}$, suppose that it has no least element. Use strong induction to prove that for all $n \geq 1$, $n \notin S$.

(3) Prove that for any finite configuration of circles in the plane, the regions formed can be colored black or white in such a way that any two regions that share a common boundary arc are colored differently.

(Hints: Induct on the number of circles. For the inductive step, remove one of the circles and color the resulting smaller configuration appropriately. Then reinsert the circle and think of a good way to modify the coloring.)

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(4) Griswald has discovered a theorem. Carefully critique his theorem and proof (what works, what doesn't- point out any errors, make suggestions for improvement):

Theorem 0.3. *For all $n \geq 1$,*

$$3 + 3^2 + \cdots + 3^n = \frac{3^{n+1} + 1}{2}.$$

Proof. Suppose, for some $n \geq k$ that

$$3 + 3^2 + \cdots + 3^k = \frac{3^{k+1} + 1}{2}.$$

Then

$$3 + 3^2 + \cdots + 3^k + 3^{k+1} = \frac{3^{k+1} + 1}{2} + 3^{k+1} = \frac{3^{k+1} + 1 + 2 \cdot 3^{k+1}}{2}$$

$$= \frac{3 \cdot 3^{k+1} + 1}{2} = \frac{3^{k+2} + 1}{2},$$

which proves the formula in the case $n = k + 1$. By induction, it holds for all $n \geq 1$. □