MAT 261 FINAL EXAM REVIEW PROBLEM SOLUTIONS

The final exam is this Wednesday, December 13, at 9:30 am in **101 Bennett**.

Finals Week Office Hours: Monday 9:30-11:00, Tuesday 10-12.

(1) Let $A = \{1, 2\}$. Find all relations on A. For each relation, determine whether it is reflexive, symmetric, transitive, antisymmetric, an equivalence relation, and/or a function from A to A.
Solution: A relation on A is a subset of

$$A \times A = \{(1,1),(1,2),(2,1),(2,2)\}.$$

Since $|A \times A| = 4$, $|\mathcal{P}(A \times A)| = 2^4 = 16$. So there are sixteen relations on A:

R	Ref?	Symm?	Trans?	Antisymm?	Eq. Rel	Fcn	comment
$\emptyset$	N	Y	Y	Y	N	N	
$\{(1,1)\}$	N	Y	Y	Y	Y	N	
$\{(1,1),(1,2)\}$	N	N	Y	Y	N	N	
$\{(1,1),(1,2),(2,2)\}$	Y	N	Y	Y	N	N	
$\{(1,1),(1,2),(2,1)\}$	N	Y	N	N	N	N	2R1 and 1R2 but not 2R2
$\{(1,1),(1,2),(2,1),(2,2)\}$	Y	Y	Y	N	Y	N	
$\{(1,1),(2,1)\}$	N	N	Y	Y	N	Y	
$\{(1,1),(2,1),(2,2)\}$	Y	N	Y	Y	N	N	
$\{(1,1),(2,2)\}$	Y	Y	Y	Y	Y	Y	the equality relation
$\{(1,2)\}$	N	N	Y	Y	N	N	
$\{(1,2),(2,1)\}$	N	Y	N	N	N	Y	
$\{(1,2),(2,2)\}$	N	N	Y	Y	N	Y	
$\{(1,2),(2,1),(2,2)\}$	N	Y	N	N	N	N	
$\{(2,1)\}$	N	N	Y	Y	N	N	
$\{(2,1),(2,2)\}$	N	N	Y	Y	N	N	
$\{(2,2)\}$	N	Y	Y	Y	N	N	

(2) Define a relation on $\mathbf{Z}$ by $m \sim n$ to mean that $m + 2n$ is divisible by 3. Prove that $\sim$ is an equivalence relation, and find [7].
Solution:
Noting that $m + 2m = 3m$ is divisible by 3 for any $m \in \mathbf{Z}$, we see that $m \sim m$, so the relation is reflexive.
Suppose $m \sim n$. Then $m + 2n = 3k$ for some $k \in \mathbf{Z}$. Hence

$$n + 2m = n + 2(3k - 2n) = -3n + 6k = 3(-n + 2k)$$

is divisible by 3. Hence $n \sim m$. Thus the relation is symmetric.
Now suppose $m \sim n$ and $n \sim r$. Write $m + 2n = 3k$ and $n + 2r = 3t$ for integers k, t. Then

$$m + 2r = (3k - 2n) + (3t - n) = 3k - 3n + 3t = 3(k - n + t)$$

is divisible by 3. Hence $m \sim r$, which shows that the relation is transitive. This proves that the relation is an equivalence relation. Note that

$$\begin{aligned}[7] &= \{m \in \mathbf{Z} |\, m + 14 \in 3\mathbf{Z}\} = -14 + 3\mathbf{Z} \\ &= 1 + 3\mathbf{Z} = \{\dots, -5, -2, 1, 4, 7, \dots\}.\end{aligned}$$

(3) Let $X = \{1, 2, 3, 4\}$, and consider the partition $\mathcal{F} = \{\{1, 2\}, \{3, 4\}\}$. Find the unique relation R on X for which $X/R = \mathcal{F}$.
Solution:

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3)\}.$$

(4) In $\mathbf{Z}/8\mathbf{Z}$, find the multiplicative inverse of [5]. Explain why [6] does not have a multiplicative inverse.
Solution:
Since $5^2 = 25 \equiv 1 \mod 8$, we see that $[5] * [5] = [25] = [1]$ in $\mathbf{Z}/8/Z$. So [5] is its own multiplicative inverse. It is interesting to note that the quadratic equation $X^2 = [1]$ has more than two solutions in $\mathbf{Z}/8\mathbf{Z}$: [1], [3], [5], [7] all satisfy the equation!
Given $[a] \in \mathbf{Z}/8\mathbf{Z}$, it has a multiplicative inverse $[b]$ only if $ab \equiv 1 \mod 8$, i.e. $ab = 1 + 8k$ for some integer k. In particular, ab is odd, and hence a is odd. This explains why the equivalence class of the even number 6 is not an invertible element of $\mathbf{Z}/8\mathbf{Z}$.

(5) Suppose $f : A \to B$ and $g : B \to C$. Suppose f and g are onto. Prove that $g \circ f$ is onto.
Solution: Let $c \in C$. Since g is onto, there exists $b \in B$ such that $g(b) = c$. Since f is onto, there exists $a \in A$ such that $f(a) = b$. Now $g \circ f(a) = g(f(a)) = g(b) = c$. Hence $g \circ f$ is onto.

(6) Suppose $f : A \to B$ and $g : B \to C$. Suppose f and g are one-to-one. Prove that $g \circ f$ is one-to-one.
Solution: Suppose $g \circ f(x) = g \circ f(y)$ for some $x, y \in A$. Then $g(f(x)) = g(f(y))$. Since g is 1-1, it follows that $f(x) = f(y)$. Since f is 1-1, it follows that $x = y$. Therefore $g \circ f$ is 1-1.

(7) Prove that $\mathbf{Z}$ and $\mathbf{Q}$ are countable.
Solution: There are many ways to do each of these. We know that a union of countably many countable sets is countable. Notice that $\mathbf{Z} = -\mathbf{N} \cup \{0\} \cup \mathbf{N}$ is the union of three countable sets. Hence $\mathbf{Z}$ is countable.

Likewise, for each $n \in \mathbf{N}$, define $A_n = \{\frac{m}{n} | m \in \mathbf{Z}\} \subseteq \mathbf{Q}$. Notice that A_n is countable since $A_n \sim \mathbf{Z}$, and $\mathbf{Z}$ is countable. (The map $m \mapsto m/n$ is a bijection from $\mathbf{Z}$ to A_n.) Every rational number can be expressed as a fraction whose denominator is a positive integer. Thus $\mathbf{Q} = \bigcup_{n \in \mathbf{N}} A_n$ is a union of countably many countable sets. Hence $\mathbf{Q}$ is countable.

(8) Prove that $2\mathbf{Z}$ is countable. Prove that the set of integer cubes $C = \{n^3 | n \in \mathbf{Z}\}$ is countable.

Define $f : \mathbf{Z} \longrightarrow 2\mathbf{Z}$ by $f(m) = 2m$. Since $\mathbf{Z}$ is countable and f is a surjection, $2\mathbf{Z}$ is countable by Theorem B. (In fact f is a bijection and $\mathbf{Z} \sim 2\mathbf{Z}$.)

Define $f : \mathbf{Z} \longrightarrow C$ by $f(n) = n^3$. Since f is onto and $\mathbf{Z}$ is countable, it follows by Theorem B that C is countable. (In fact f is a bijection and $\mathbf{Z} \sim C$.)

(9) Prove that any subset of a countable set is countable.
Solution: Let A be a countable set, and let $B \subseteq A$. The identity map $i_B : B \longrightarrow A$ (defined by $i_B(b) = b$) is an injection from B into the countable set A. Therefore by Theorem B, B is countable.

(10) Prove that for any set A, there is an injection $f : A \longrightarrow \mathcal{P}(A)$.

Proof. An example of such a function is given by $f(a) = \{a\}$ for $a \in A$. If $f(a) = f(b)$ for some $a, b \in A$, then $\{a\} = \{b\}$, which means that $a = b$, so f is one-to-one. □

(11) Prove that for any set A, there is no surjection $f : A \longrightarrow \mathcal{P}(A)$.

Proof. Suppose $f : A \longrightarrow \mathcal{P}(A)$ is any function. Let

$$B = \{x \in A | x \notin f(x)\}.$$

Then $B \in \mathcal{P}(A)$.

Suppose there exists $y \in A$ such that $f(y) = B$. Note that either $y \in B$ or $y \notin B$. If $y \in B$, then by the definition of B, $y \notin f(y)$. Since $f(y) = B$, this means that $y \notin B$, which is a contradiction.

On the other hand, suppose $y \notin B$. Then since $B = f(y)$, this means that $y \notin f(y)$, which means exactly that $y \in B$, another contradiction. To summarize, the assumption that $f(y) = B$ for some $y \in A$ leads to a contradiction. Hence $f(x) \neq B$ for all $x \in A$, which proves that f is not onto. □

(12) Analyze the following proof that $\mathbf{Z}$ is countable. The identity map is an injection from $\mathbf{N} \longrightarrow \mathbf{Z}$. Since there is an injection from $\mathbf{N}$ into $\mathbf{Z}$, $\mathbf{Z}$ is countable.
Solution: The logic is flawed. The existence of an injection from $\mathbf{N}$ to a

set A does not imply that A is countable. Consider the map $f : \mathbf{N} \longrightarrow \mathbf{R}$ defined by $f(n) = n$. Then f is 1-1, but $\mathbf{R}$ is uncountable.

(13) Prove that $\mathbf{R}^+ \sim \mathbf{R}$ by finding an explicit bijection between them.
Solution:
Define a map $f : \mathbf{R} \longrightarrow \mathbf{R}^+$ by $f(x) = e^x$. Define $g : \mathbf{R}^+ \longrightarrow \mathbf{R}$ by $g(y) = \ln(y)$. Then for any $x \in \mathbf{R}$, $g(f(x)) = \ln(e^x) = x$, and for any $y \in \mathbf{R}^+$, $f(g(y)) = e^{\ln y} = y$. This shows that f is invertible with inverse g. We know that a function is invertible if and only if it is a bijection. This proves that $\mathbf{R} \sim \mathbf{R}^+$.

(14) (Room for one more!) Let A be an infinite set, and suppose $x \notin A$. Prove that $A \sim (A \cup \{x\})$.
Solution: By HW11, A has a denumerable subset $B = \{b_1, b_2, \ldots\}$. Define a map $f : A \longrightarrow A \cup \{x\}$ by

$$f(a) = \begin{cases} a & \text{if } a \notin B \\ b_{n-1} & \text{if } a = b_n \in B \text{ for } n > 1 \\ x & \text{if } a = b_1. \end{cases}$$

We show that f is onto. Given $c \in A \cup \{x\}$, there are three possibilities: either $c \notin B \cup \{x\}$, or $c \in B$, or $c = x$. In the first case, $f(c) = c$. In the second case, $c = b_n$ for some $n \in \mathbf{N}$, and $f(b_{n+1}) = b_n = c$. In the last case, $f(b_1) = x = c$. Hence f is onto.

To show f is 1-1, suppose $f(a) = f(b)$ for some $a, b \in A$. There are three cases: $f(a) \notin B \cup \{x\}$, $f(a) \in B$, and $f(a) = x$. In the first case, $a = f(a) = f(b) = b$. If $f(a) \in B$, then $f(a) = f(b) = b_n$ for some n. From the definition of f, the only possibility is that $a = b = b_{n+1}$. If $f(a) = x$, then again from the definition of f, we see that $a = b_1 = b$. Hence in all cases, $a = b$, so f is 1-1.

(15) (Room for lots more!) Let A be an infinite set, and let B be any finite set with $B \cap A = \emptyset$. Prove that $A \sim (A \cup B)$.

Proof. We induct on $n = |B|$. In the base case where $n = 1$, we let x be the unique element of B. By the previous problem, there is a bijection from A onto $A \cup B$. Suppose for some $n \geq k$ that $A \sim (A \cup B)$ for all sets B of size k with $A \cap B = \emptyset$. Now let $B = \{b_1, b_2, \ldots, b_{k+1}\}$ be a set of size $k + 1$ which is disjoint from A. By the inductive hypothesis,

$$A \sim A \cup \{b_1, \ldots, b_k\}.$$

By the base case,

$$A \cup \{b_1, \ldots, b_k\} \sim [A \cup \{b_1, \ldots, b_k\}] \cup \{b_{k+1}\} = A \cup B.$$

By transitivity of set equivalence,

$$A \sim A \cup B.$$

Finally by induction, the original statement holds for any finite set B. $\square$

Every induction proof needs to have:

(a) Clear statement that you are using induction.
(b) Clear statement and proof of the base case.
(c) Clear statement of the inductive hypothesis.
(d) Concluding statement.

(16) Prove that the real interval $(0,1)$ is equivalent (in the sense of equivalence of sets) to $(-\frac{\pi}{2}, \frac{\pi}{2})$.
Solution: We first translate the interval $(0,1)$ left by $1/2$ to the interval $(-1/2, 1/2)$. (The function that does this is $f(x) = x - 1/2$.) Then we dilate by multiplying everything by π to expand $(-1/2, 1/2)$ to the interval $(-\pi/2, \pi/2)$. Explicitly, define $f : (0,1) \longrightarrow (-\pi/2, \pi/2)$ by

$$f(x) = \pi(x - \frac{1}{2}).$$

First, it is important to point out that $f(x)$ does indeed belong to $(-\pi/2, \pi/2)$, since $x - 1/2 \in (-1/2, 1/2)$. Next, if $f(a) = f(b)$ for some $a, b \in (0,1)$, then $\pi(a - 1/2) = \pi(b - 1/2)$. Dividing by π and then adding $1/2$, gives $a = b$. Hence f is 1-1. Now given $c \in (-\pi/2, \pi/2)$,

$$f(\tfrac{c}{\pi} + \tfrac{1}{2}) = \pi(\frac{c}{\pi} + \frac{1}{2} - \frac{1}{2}) = c,$$

so f is onto. Hence f is a bijection.

(17) Prove that $(0,1) \sim \mathbf{R}$.
Solution: We know that $(0,1) \sim (-\pi/2, \pi/2)$ by the previous problem. The function $f : (-\pi/2, \pi/2) \longrightarrow \mathbf{R}$ defined by $f(x) = \tan(x)$ is a bijection, since it is invertible, with inverse $\arctan(x)$. Hence $(-\pi/2, \pi/2) \sim \mathbf{R}$, and by transitivity of equivalence, $(0,1) \sim \mathbf{R}$.

(18) Let $n \geq 2$, and let $A_1, \ldots, A_n$ be sets in some universe. Using induction, prove the following generalization of DeMorgan's Law:

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$

Proof. We induct on n. For the base case, suppose $n = 2$. By DeMorgan's Law, we know that

$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c,$$

as needed. For the inductive step, suppose for some $k \geq 2$ that for any collection of k sets $A_1, \ldots A_k$,

$$\left(\bigcap_{i=1}^{k} A_i\right)^c = \bigcup_{i=1}^{k} A_i^c.$$

Now suppose we are given $k+1$ sets $A_1, \ldots, A_{k+1}$. Then

$$\left(\bigcap_{i=1}^{k+1} A_i\right)^c = \left(\left[\bigcap_{i=1}^{k} A_i\right] \cap A_{k+1}\right)^c = \left[\bigcap_{i=1}^{k} A_i\right]^c \cup A_{k+1}^c$$

by DeMorgan's Law (the base case). Now applying the inductive hypothesis, the above is

$$= \left[\bigcup_{i=1}^{k} A_i^c\right] \cup A_{k+1}^c = \bigcup_{i=1}^{k+1} A_i^c,$$

which proves the desired identity in the case where $n = k+1$. By induction, the identity holds for all $n \geq 2$. $\square$

(19) Suppose there is an injection $f : A \longrightarrow B$ and a surjection $g : A \longrightarrow B$. Prove that $A \sim B$. (Hint: Use the Schröder-Bernstein theorem.)

Proof. We will use g to define an injection $h : B \longrightarrow A$ as follows. For each $b \in B$, we may choose some $a \in A$ such that $g(a) = b$ since g is onto. There may well be many such elements a, but we select one of them and denote it by $h(b)$. (We are using the *axiom of choice* here, in case you care.) The resulting function h is an injection. Indeed, if $h(x) = h(y)$ for some $x, y \in B$, then $x = g(h(x)) = g(h(y)) = y$.

Now we have injections from A to B and from B to A. By the SB Theorem, $A \sim B$. $\square$

(20) Suppose A and B are equivalent sets: $A \sim B$. Prove that $\mathcal{P}(A) \sim \mathcal{P}(B)$.

Proof. Let $f : A \longrightarrow B$ be a bijection. Define a map $F : \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$ by

$$F(X) = f(X) = \{f(x) \,|\, x \in X\}.$$

In words, we associate to a subset $X \subseteq A$ the subset $F(X) \subseteq B$ obtained by applying f to each element of X. We claim that F is a bijection.

Suppose $F(X) = F(Y)$ for some $X, Y \in \mathcal{P}(A)$. We need to show that $X = Y$. Given any $x \in X$, we know that $f(x) \in F(X) = F(Y)$, so $f(x) = f(y)$ for some $y \in Y$. Since f is one-to-one, it follows that $x = y$, and in particular, $x \in Y$. Hence $X \subseteq Y$. Similarly, $Y \subseteq X$, so $X = Y$. This proves that F is one-to-one.

Now given any subset $C \subseteq \mathcal{P}(B)$, let

$$X = f^{-1}(C) = \{x \in A \,|\, f(x) \in C\}.$$

Clearly $F(X) \subseteq C$. Conversely, for any $c \in C$, there exists $x \in A$ such that $f(x) = c$ since f is onto. This means that $x \in X$, so $c = f(x) \in F(X)$. Hence $F(X) = C$, so F is onto.

In summary, F is a bijection, so $\mathcal{P}(A) \sim \mathcal{P}(B)$. $\square$