MAT 261 FINAL EXAM REVIEW

The final exam is Wednesday, December 13, from 9:30 - 11:30 am in **101 Bennett Hall**. (THIS IS ACROSS FROM OUR USUAL CLASSROOM!)

Finals Week Office Hours: Tuesday 10-12.

The final will consist of two parts. The first part will cover the same material covered on the midterm. The second part will cover everything else. So study the midterm review material and also the midterm itself.

What follows is a review of the topics covered after the midterm.

1. Relations

Let A and B be sets. A *relation* from A to B is a subset $R \subseteq A \times B$. In the special case where $A = B$, a subset $R \subseteq A \times A$ is a *relation on* A. We typically write aRb to signify that $(a, b) \in R$.

Example: The *equality relation* on A is the set

$$i_A = \{(x, x) | \, x \in A\} \subseteq A \times A.$$

In this case, xRy if and only if $x = y$. This is why it is called the equality relation.

Let R be a relation on A. Know what it means for R to be: reflexive, symmetric, transitive, antisymmetric, a function.

2. Equivalence relations

If R is a relation on A, then it is an *equivalence relation* if it is reflexive, symmetric and transitive.

Let R be an equivalence relation on A. Then the *equivalence class* of an element $a \in A$ is the subset

$$[a] = \{y \in A | \, yRa\} \subseteq A.$$

An element of $[a]$ is a *representative* for $[a]$.

We let A/R ("A mod R") denote the set of equivalence classes.

3. Partitions

Let A be a set, and let $\mathcal{F} \subseteq \mathcal{P}(A)$ be a collection of subsets of A. We say that $\mathcal{F}$ is a *partition* of A if

(1) $A = \bigcup_{C \in \mathcal{F}} C$
(2) $\mathcal{F}$ is *pairwise disjoint*, i.e., for $B, C \in \mathcal{F}$, $B \cap C = \emptyset$ unless $B = C$.
(3) For all $C \in \mathcal{F}$, $C \neq \emptyset$.

In this case, we call the sets $C \in \mathcal{F}$ the *cells* of the partition.

A partition is really the same concept as an equivalence relation. This is the content of the following theorem, proven in class.

Theorem 3.1. *If R is an equivalence relation on a set A, then A/R is a partition of A. Conversely, given any partition $\mathcal{F}$ of A, there is an equivalence relation R on A such that $A/R = \mathcal{F}$.*

4. Integers mod m

Fix an integer m, and define a relation on $\mathbf{Z}$ by

$$x \equiv y \mod m$$

to mean that $x - y$ is divisible by m. Equivalent ways to say/write this:

(1) $x - y \in m\mathbf{Z}$
(2) $m|(x - y)$
(3) There exists $k \in \mathbf{Z}$ such that $x - y = km$.
(4) There exists $k \in \mathbf{Z}$ such that $x = y + km$.

More formally, the relation is the set

$$R_m = \{(x, y) \in \mathbf{Z}^2 |\, m|(x - y)\}.$$

The equivalence class of an integer a is

$$[a] = \{x \in \mathbf{Z}|\, x \equiv a \mod m\}$$
$$= \{x \in \mathbf{Z}|\, x = a + mk \text{ for some } k \in \mathbf{Z}\} = a + m\mathbf{Z}.$$

Rather than using the usual notation $\mathbf{Z}/R_m$ for the set of equivalence classes, we write it as $\mathbf{Z}/m\mathbf{Z}$. This is the set of integers modulo m.

We *defined* the following arithmetic operations in $\mathbf{Z}/m\mathbf{Z}$:

$$[a] + [b] = [a + b], \qquad [a][b] = [ab].$$

We showed that these definitions do not rely on the particular choice a, b of representatives for $[a]$ and $[b]$.

5. Functions

Given sets A and B, a *function* (or *map*) from A to B is a relation $f \subseteq A \times B$ with the property that for all $x \in A$, there exists a unique $b \in B$ such that $(x, y) \in f$. We use the shorthand $f(x) = y$ to signifiy $(x, y) \in f$. The notation $f : A \longrightarrow B$ signifies that f is a function from A to B.

Definitions to know: function, domain, codomain, image (range is a synonym for image), one-to-one, onto, injection, surjection, bijection, invertible.

Theorem 5.1. *A function $f : A \longrightarrow B$ is invertible if and only if it is a bijection.*

To prove that a given function $f : A \longrightarrow B$ is onto: Let $b \in B$. Produce an appropriate $a \in A$ (on your scratch work) and SHOW that $f(a) = b$.
DO NOT: start with the equation $f(a) = b$ and solve for a (though it may be appropriate to do so on scratch work).

To prove that f is not onto: Produce an appropriate $b \in B$ and explain why $f(x) \neq b$ for all $x \in A$.

To prove that f is one-to-one: Suppose, for some $x, y \in A$, that $f(x) = f(y)$. Prove that $x = y$.

To prove that f is not one-to-one: Produce two elements $x \neq y$ in A for which $f(x) = f(y)$.

Theorem 5.2 (Useful Fact)**.** *If A and B are* **finite** *sets of the* **same size***, and $f : A \longrightarrow B$, then the following statements are equivalent:*

(1) *f is onto*
(2) *f is one-to-one*
(3) *f is a bijection.*

6. Cardinality

Two sets A and B are *equivalent* (denoted $A \sim B$) if there is a bijection $f : A \longrightarrow B$. In this case we say A and B have the same *cardinality* and write $|A| = |B|$. It means that they have the same size. This is an equivalence relation on sets.

We let $\aleph_0$ denote the cardinality of $\mathbf{N}$, and c denotes the cardinality of $\mathbf{R}$.

A set A is *denumerable* if $A \sim \mathbf{N}$, i.e. $|A| = \aleph_0$.

A set A is *countable* if it is either finite or denumerable. Make sure to memorize the following theorems, which are extremely useful for determining whether or not a given set is countable.

Theorem 6.1 (Theorem B)**.** *Let A be a set. Then the following statements are equivalent:*

1. *A is countable*
2. *There exists a countable set B and a surjection $f : B \longrightarrow A$.*
3. *There exists a countable set C and an injection $g : A \longrightarrow C$.*

Theorem 6.2. *If A and B are countable, then so are $A \cup B$ and $A \times B$.*

Theorem 6.3 (Theorem C)**.** *The union of a countable collection of countable sets is countable.*

Theorem 6.4. $\mathbf{Z}$*,* $\mathbf{Q}$ *and* $\overline{\mathbf{Q}}$ *(the set of all algebraic complex numbers) are all countable.* $\mathbf{R}$ *is uncountable.*

We write $|A| \leq |B|$ if there is an injection from A to B.

We write $|A| < |B|$ if $|A| \leq |B|$, but $A \nsim B$. In other words, there is an injection from A to B, but no bijection from A to B. (It is known, but we did not prove, that $|A| < |B|$ if and only if there is no surjection from A to B.)

The Schröder-Bernstein Theorem states that if $|A| \leq |B|$ and $|B| \leq |A|$ then $A \sim B$.

Cantor's Continuum Hypothesis asserts that there is no set X whose cardinality lies strictly between $\aleph_0$ and c. This cannot be proven or disproven within the usual axioms of set theory.

7. Sample Problems

Be sure to also study the midterm and the sample problems for the midterm. You may also be asked for definitions of terms mentioned above. Make sure you know them well. No notes or calculators are allowed. Also fair game: HW and midterm problems!

(1) Let $A = \{1, 2\}$. Find all relations on A. For each relation, determine whether it is reflexive, symmetric, transitive, antisymmetric, an equivalence relation, and/or a function.

(2) Define a relation on $\mathbf{Z}$ by $m \sim n$ if and only if $m + 2n$ is divisible by 3. Prove that $\sim$ is an equivalence relation, and find $[7]$.

(3) Let $X = \{1, 2, 3, 4\}$, and consider the partition $\mathcal{F} = \{\{1, 2\}, \{3, 4\}\}$. Find the unique relation R on X for which $X/R = \mathcal{F}$.

(4) In $\mathbf{Z}/8\mathbf{Z}$, find the multiplicative inverse of $[5]$. Explain why $[6]$ does not have a multiplicative inverse.

(5) Suppose $f : A \to B$ and $g : B \to C$. Suppose f and g are onto. Prove that $g \circ f$ is onto.

(6) Suppose $f : A \to B$ and $g : B \to C$. Suppose f and g are one-to-one. Prove that $g \circ f$ is one-to-one.

(7) Prove that $\mathbf{Z}$ and $\mathbf{Q}$ are countable.

(8) Prove that $2\mathbf{Z}$ is countable. Prove that the set of integer cubes $\{n^3 | \, n \in \mathbf{Z}\}$ is countable.

(9) Prove that any subset of a countable set is countable.

(10) Prove that for any set A, there is an injection $f : A \longrightarrow \mathcal{P}(A)$.

(11) Prove that for any set A, there is no surjection $f : A \longrightarrow \mathcal{P}(A)$. (The upshot of these two problems is that $|A| < |\mathcal{P}(A)|$ for all sets A.)

(12) Analyze the following proof that $\mathbf{Z}$ is countable. The identity map is an injection from $\mathbf{N} \longrightarrow \mathbf{Z}$. Since there is an injection from $\mathbf{N}$ into $\mathbf{Z}$, $\mathbf{Z}$ is countable.

(13) Prove that $\mathbf{R}^+ \sim \mathbf{R}$ by finding an explicit bijection between them. (Recall that $\mathbf{R}^+ = (0, \infty)$.)

(14) (Room for one more!) Let A be an infinite set, and suppose $x \notin A$. Prove that $A \sim (A \cup \{x\})$. (This was on HW 12.)

(15) (Room for lots more!) Let A be an infinite set, and let B be any finite set with $B \cap A = \emptyset$. Prove that $A \sim (A \cup B)$. (Hint: Induct on $n = |B|$, and use the previous problem.)

(16) Prove that the real interval $(0, 1)$ is equivalent (in the sense of equivalence of sets) to $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(17) Prove that $(0, 1) \sim \mathbf{R}$. (Think about the function $\tan x$ and use the previous problem.)

(18) Let $n \geq 2$, and let $A_1, \ldots, A_n$ be sets in some universe. Using induction, prove the following generalization of DeMorgan's Law:

$$\left(\bigcap_{i=1}^{n} A_i \right)^c = \bigcup_{i=1}^{n} A_i^c.$$

(19) Suppose there is an injection $f : A \longrightarrow B$ and a surjection $g : A \longrightarrow B$. Prove that $A \sim B$. (Hint: Use the Schröder-Bernstein theorem.)

(20) Suppose A and B are equivalent sets: $A \sim B$. Prove that $\mathcal{P}(A) \sim \mathcal{P}(B)$.