MAT 136 MIDTERM 2 REVIEW PROBLEM SOLUTIONS

Midterm 2 is Thurs, November 16. Please arrange a 2-hour block of time.

(1) Prove that the function $f(x) = x^2$ is continuous at the point 2.

Proof. Fix $\epsilon > 0$. Let $\delta = \min(1, \frac{\epsilon}{5})$. Then if $|x - 2| < \delta$, we see that

$$|x + 2| = |x - 2 + 4| \leq |x - 2| + 4 < \delta + 4 \leq 5.$$

Therefore for such x,

$$|f(x) - f(2)| = |x^2 - 4| = |x - 2||x + 2| \leq 5|x - 2| < 5\delta \leq 5\frac{\epsilon}{5} = \epsilon.$$

Hence f is continuous at 2. □

(2) (Problem 6-15) Suppose f is a function and $\lim_{x\to a^+} f(x) = f(a) < 0$. Prove that there exists $\delta > 0$ such that $f(x) < 0$ for all x satisfying $a \leq x < a + \delta$. (This is a one-sided version of Theorem 6-3.)

Proof. By definition, $\lim_{x\to a^+} f(x) = f(a)$ means that for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $0 < x - a < \delta$.

Apply this with $\epsilon = |f(a)|$. The condition $0 < x - a < \delta$ is the same as $a < x < a + \delta$, and for such x, $f(x)$ is within $|f(a)|$ of $f(a)$, so in particular, $f(x) < f(a) + |f(a)| = 0$ since $f(a) < 0$. □

(3) Prove Theorem 7-1. The theorem states: If f is continuous on a closed interval $[a, b]$, and $f(a) < 0 < f(b)$, then there exists $c \in [a, b]$ for which $f(c) = 0$.

Proof. We gave Spivak's proof in class. The following proof is slightly different (easier?).

Let

$$A = \{x \in [a, b] : f(x) < 0\}.$$

The set A is nonempty because it was given that $f(a) < 0$, and hence $a \in A$. For any $x \in A$, $x \leq b$. This shows that A is bounded above by b. Therefore by the completeness axiom of the real numbers (P13), A has a least upper bound, c.

Editorial comment: look carefully at the statement of (P13). It has nothing to do with functions! Students are sometimes confused about this.

First, we claim that $a < c < b$. Note that for some $\delta > 0$, $f(x) < 0$ for all $x \in [a, a+\delta)$, that is, every element of $[a, a+\delta)$ belongs to A. Indeed, we showed exactly this in the previous problem. We are using the continuity of f in a crucial way here. It follows that $c \geq a+\delta$, and in particular $c > a$.

Similarly, $c < b$. Indeed, we know that $c \leq b$ by definition of supremum, since b is an upper bound for A. Because $f(b) > 0$, there exists $\varepsilon > 0$ such that $f(x) > 0$ for all $x \in (b - \varepsilon, b]$ (by an analog of the previous problem). For any $a \in A$, $a \leq b$, and as just shown, there exist no elements of A in $(b - \varepsilon, b]$. Therefore $a \leq b - \varepsilon$, i.e. $b - \varepsilon$ is an upper bound for A. By the definition of supremum, it follows that $c \leq b - \varepsilon < b$.

We have established that $a < c < b$, and we now show that $f(c) = 0$.

First, suppose $f(c) < 0$. By Theorem 6-3, there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in (c - \delta, c + \delta)$. (Because $c < b$, we can take δ small enough that $c + \delta < b$). Let $x_0 \in (c, c + \delta)$. Then $f(x_0) < 0$, i.e. $x_0 \in A$, and $c < x_0$, a contradiction since c is an upper bound for A. Therefore we must conclude that $f(c) \geq 0$.

Now suppose $f(c) > 0$. By Theorem 6-3, there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta)$. On the other hand, because $c = \sup(A)$, there exists $a \in A$ such that $a \in (c - \delta, c]$. For such a, $f(a) > 0$ by the above, while $f(a) < 0$ since $a \in A$. This is a contradiction.

Finally, we may conclude that $f(c) = 0$. □

(4) (Spivak, Chapter 8, problem 14) Consider a nested sequence of closed intervals:

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \cdots$$

Prove that there is a point c which belongs to all of the intervals.

Proof. Let $A = \{a_1, a_2, \ldots\}$ be the set of all of the above left endpoints. For all $n \geq 1$, we are given that $[a_n, b_n] \subseteq [a_1, b_1]$. In particular, this means that $a_n \leq b_1$ for all n. This shows that b_1 is an upper bound for the set A. Since A is nonempty and bounded above, it has a least upper bound c.

We now show that $c \in [a_n, b_n]$ for all n. Fix such $n \geq 1$. Clearly $a_n \leq c$ since $a_n \in A$ and c is an upper bound for A. We just need to show that $c \leq b_n$. Suppose to the contrary that $c > b_n$. Then since c is the least upper bound of A, there exists an element $a_m \in A$ for which $b_n < a_m \leq c$. But this would imply that the intervals $[a_n, b_n]$ and $[a_m, b_m]$ are disjoint, in violation of the given information that one is contained in the other. By this contradiction, we conclude that $c \leq b_n$, as needed. □

(5) Prove Fermat's Theorem, which states that if c is either a local maximum or a local minimum of a function f, then c is a critical point, i.e., either $f'(c) = 0$ or $f'(c)$ does not exist.

Proof. Suppose f is differentiable at c. We must show that $f'(c) = 0$. Let's consider the case where c is a local minimum. This means (by definition!) that there exists $\delta > 0$ such that $f(x) \geq f(c)$ whenever $|x - c| < \delta$. Since f is differentiable at c,

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}.$$

For all x satisfying $c < x < c + \delta$, the difference quotient above has nonnegative numerator and positive denominator. So $f'(c) \geq 0$. At the same time,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

When x satisfies $c - \delta < x < c$, the above difference quotient has nonnegative numerator but negative denominator. So $f'(c) \leq 0$. Putting these together, we see that $f'(c) = 0$. □

(6) Suppose $f'(x) = 0$ for all x in an interval. Prove that f is constant on the interval.

Proof. Fix any point a in the interval. We will show that $f(x) = f(a)$ for all x in the interval. This is obvious if $x = a$, so let's suppose $x \neq a$. By the Mean Value Theorem, there exists c between a and x such that

$$f'(c) = \frac{f(x) - f(a)}{x - a}.$$

By hypothesis, $f'(c) = 0$. Therefore the above fraction is equal to 0, so the numerator must be 0. This proves that $f(x) = f(a)$, so f is constant. □

For a lot more, click on: MAT-126-old-exams.

I particularly suggest doing the following (solutions are found on the above website).

Sample1: 1, 5, 6, 8, 9, 10

F16 MT1: 3, 4, 6, 7

Sample 2: 1 (ignore all log and inverse trig problems for now), 3, 4, 6

F16 MT2: 1, 2, 4, 6

F14 MT2: 1, 2, 3, 5, 6