MATH 126

MIDTERM 1 REVIEW PROBLEMS SOLUTIONS

October 3, 2017

1. (10 pts) Prove that $\sqrt{3}$ is irrational.

Suppose, to the contrary, that $\sqrt{3}$ is rational. Then

$$\sqrt{3} = \frac{a}{b}$$

for some integers a, b with $b \neq 0$. We may assume that the fraction is reduced, so that a and b have no common prime factors. Squaring both sides and cross-multiplying, we see that

$$3b^2 = a^2.$$

In the prime decomposition of a^2, each prime occurs an even number of times. But in the prime decomposition of $3b^2$, the prime 3 occurs an odd number of times. This is a contradiction. Therefore $\sqrt{3}$ is irrational.

2. (10 pts) A set S of real numbers is *dense* if, for every pair of real numbers $a < b$, there is an element of S between a and b. For example, $\mathbf{Q}$ is dense because we can find a number between a and b that has a finite decimal expansion, which means it is of the form $\frac{n}{10^m}$ and hence rational. Using the fact that $\mathbf{Q}$ is dense, show that the set of irrational numbers is also dense.

Proof. Given any real numbers $a < b$, it follows that $a + \sqrt{3} < b + \sqrt{3}$. Because $\mathbf{Q}$ is dense, there is a rational number q satisfying

$$a + \sqrt{3} < q < b + \sqrt{3},$$

which implies

$$a < q - \sqrt{3} < b.$$

Noting that $q - \sqrt{3}$ is irrational, we conclude that the irrational numbers are dense. (Remark: If $x = q - \sqrt{3}$ were rational, then $\sqrt{3} = q - x \in \mathbf{Q}$, contradicting problem 1.) $\square$

3. (10 pts) Give an example of each of the following, if possible. If not possible, then briefly state why.

(a) Two irrational numbers whose product is rational: $\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 1 \in \mathbf{Q}$.

(b) Two irrational numbers whose product is irrational: $\sqrt{2}\sqrt{3} = \sqrt{6}$ is irrational by HW.

(c) A function defined on all the real numbers but continuous nowhere. Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ -1 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Then for any $a \in \mathbf{R}$, $\lim_{x \to a} f(x)$ does not exist. Indeed, for any $\delta > 0$, there is an irrational number $y \in (a-\delta, a)$ and a rational number $x \in (a-\delta, a)$, by Problem 2. Since $f(x) = 1$ and $f(y) = -1$, these values cannot both be within $\varepsilon = \frac{1}{3}$ of any limit L.

(d) A function $f : \mathbf{N} \longrightarrow \mathbf{N}$ such that $f \circ f = f$, but $f(n) \neq n$ for all n.

Not possible. Let $n = f(1)$. Then $f(n) = f(f(1)) = f(1) = n$.

4. (12 pts) Prove using mathematical induction that for any $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

For the base case $n = 1$, the assertion says simply that $\frac{1}{1 \cdot 2} = \frac{1}{2}$, which is obvious. Now suppose, for some $k \geq 1$, that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}.$$

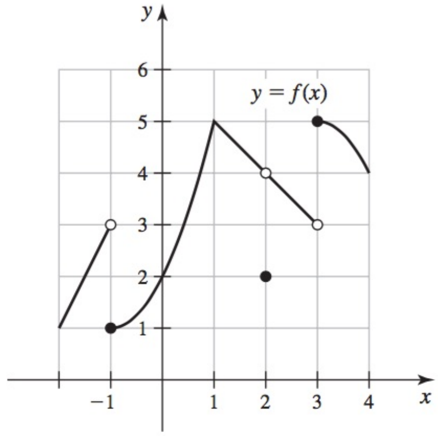
Then

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2},$$

which establishes the formula in the case $n = k+1$. By induction, it holds for all $n \geq 1$.

5. (10 pts) Let $f(x)$ be the function whose graph is shown below.



For each problem, give the value, or explain why it does not exist.

(a) $\lim_{x \to 1} f(x) = 5$.

(b) $\lim_{x \to 2} f(x) = 4$. (Remember the limit does not depend on the value $f(2)$!)

(c) $f(2) = 2$.

(d) $\lim_{x \to 3} f(x)$. Does not exist. The one-sided limits do not match up:

$$\lim_{x \to 3^+} f(x) = 5, \qquad \lim_{x \to 3^-} f(x) = 3.$$

(e) List all values of x at which f fails to be continuous. The bad values of x are: $-1, 2, 3$.

6. (12 pts) Evaluate the following limits.

(a) $\lim_{x \to 4} \frac{\sqrt{1+2x}-1}{x} = \frac{\sqrt{1+8}-1}{4} = \frac{1}{2}$.

(b) $\lim_{x \to 0} \frac{\sqrt{1+2x}-1}{x} = \lim_{x \to 0} \frac{(\sqrt{1+2x}-1)}{x} \cdot \frac{(\sqrt{1+2x}+1)}{(\sqrt{1+2x}+1)} = \lim_{x \to 0} \frac{1+2x-1}{x(\sqrt{1+2x}+1)}$

$$= \lim_{x \to 0} \frac{2x}{x(\sqrt{1+2x}+1)} = \lim_{x \to 0} \frac{2}{\sqrt{1+2x}+1} = \frac{2}{1+1} = 1.$$

(c) $\lim_{h \to 1^-} \frac{-1}{h-1}$. This is a limit of type $\frac{-1}{0}$: since there is a nonzero numerator with denominator zero, the limit will equal $\pm\infty$. When $h < 1$ is very close to 1, the expression looks like $\frac{-1}{\text{(tiny negative number)}}$, which is a large positive number. Therefore $\lim_{h \to 1^-} \frac{-1}{h-1} = \infty$.

7. (24 pts)

(a) Define what it means for $\lim_{x \to a} f(x) = L$. It means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.
MAKE SURE YOU HAVE THIS CAREFULLY MEMORIZED.

(b) Show, using the definition of limit, that $\lim_{x \to 4} \dfrac{x+1}{x+6} = \dfrac{1}{2}$.

Fix $\varepsilon > 0$. We need to find $\delta > 0$ such that $|\frac{x+1}{x+6} - \frac{1}{2}| < \varepsilon$ whenever $0 < |x - 4| < \delta$. Note that

$$\left|\frac{x+1}{x+6} - \frac{1}{2}\right| = \left|\frac{2x+2-(x+6)}{2(x+6)}\right| = \frac{|x-4|}{2|x+6|}.$$

Suppose x is within 1 unit of 4. Then $x + 6$ is within 1 unit of 10. In particular, $|x+6| = x+6 \geq 9$. Then for such x, $\frac{|x-4|}{2|x+6|} \leq \frac{|x-4|}{18}$. (We have replaced the denominator by a smaller number which results in a larger value for the fraction).

Let $\delta = \min\{1, \varepsilon\}$. Then if $|x - 4| < \delta$, x is within one unit of 4, so the above estimate is valid. So for such x,

$$\left|\frac{x+1}{x+6} - \frac{1}{2}\right| = \frac{|x-4|}{2|x+6|} \leq \frac{|x-4|}{18} < \frac{\delta}{18} \leq \frac{\varepsilon}{18} < \varepsilon.$$

This proves that the above limit is $1/2$.

(c) Show, using the definition of limit, that $\lim_{x \to 0} x\cos(\frac{1}{x}) = 0$.

Fix $\varepsilon > 0$, and let $\delta = \varepsilon$. Then if $0 < |x| < \delta = \varepsilon$, we have

$$|x\cos(\tfrac{1}{x})| \leq |x| < \varepsilon,$$

as needed. Alternatively, we could simply have applied the result of the next problem with $M = 1$:

(d) Suppose $\lim_{x \to 0} f(x) = 0$, and $g(x)$ is a bounded function (this means that there exists $M > 0$ such that $|g(x)| \leq M$ for all x). Prove using the definition of limit that

$$\lim_{x \to 0} f(x)g(x) = 0.$$

Fix $\varepsilon > 0$. Since $\lim_{x \to 0} f(x) = 0$, there exists $\delta > 0$ such that $|f(x)| < \frac{\varepsilon}{M}$ whenever $0 < |x| < \delta$. For such x,

$$|f(x)g(x)| \leq M|f(x)| < M\frac{\varepsilon}{M} = \varepsilon,$$

which proves that $\lim_{x \to 0} f(x)g(x) = 0$.

8. (10 pts) The line $y = L$ is a *horizontal asymptote* for a graph $y = f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.
Find all horizontal asymptotes for the graph of the function

$$f(x) = \frac{-3x^2 - 4x^5}{2x + x^5}.$$

As shown in class,

$$\lim_{x\to\infty} \frac{-3x^2 - 4x^5}{2x + x^5} = \lim_{x\to\infty} \frac{-4x^5}{x^5} = -4.$$

So $y = -4$ is a horizonal asymptote. The limit as $x \to -\infty$ is the same, so there is only one such asymptote.

9. (6 pts) Suppose $|x - a| < 1$. Prove that $|x + a| < 1 + 2|a|$.

By the triangle inequality,

$$|x + a| = |x - a + 2a| \leq |x - a| + |2a| < 1 + 2|a|.$$