Set 12 → Differential Calculus

a) 
\[ f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{1}{x} - \frac{1}{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{\frac{x_0 - x}{xx_0}}{x - x_0} = \lim_{x \to x_0} \frac{-1}{xx_0} = -\frac{1}{x^2}. \]

b) 
\[ f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}. \]

We know that \( f(0) = 0 \) and that \( \left| \frac{f(x)}{x} \right| \leq |x| \). So by the definition of the limit seen in class, we know that \( \lim_{x \to 0} \frac{f(x)}{x} = 0 = f'(0) \). So \( f \) is differentiable at 0.

c) Take \( f(x) = \sin \frac{x^2}{x} \). Since the numerator of \( f \) will always be between -1 and 1, we can see that \( \lim_{x \to \infty} f(x) = 0 \). However, \( f'(x) = 2 \cos x^2 - \sin x^2 \). And since \( \lim_{x \to \infty} 2 \cos x^2 \not\exists \), then \( \lim_{x \to \infty} f'(x) \not\exists \).

d) We want to apply Rolle’s Theorem, so we need a function \( f(x) \) such that \( f'(x) = a_0 + a_1 x + \ldots + a_n x^n \). So, let’s take

\[ f(x) = a_0 x + \frac{a_1 x^2}{2} + \ldots + \frac{a_n x^{n+1}}{n + 1}. \]

Then, \( f(0) = 0 \) and \( f(1) = a_0 + a_1/2 + \ldots + a_n/(n + 1) = 0 \) by hypothesis. Thus, by Rolle’s Thm, \( \exists x \in (0, 1) \) such that

\[ f'(x) = a_0 + a_1 x + \ldots + a_n x^n = 0. \]

e) Apply the Mean Value Theorem to \( f(x) = \sqrt{x} \) on the interval \([64, 66]\):

\[ \frac{\sqrt{66} - \sqrt{64}}{66 - 64} = f'(x_0) = \frac{1}{2\sqrt{x_0}} \]

for some \( x_0 \in [64, 66] \). Since \( 64 < x_0 < 81 \), we have \( 8 < \sqrt{x_0} < 9 \), so

\[ \frac{1}{2\sqrt{81}} < \frac{\sqrt{66} - 8}{2} < \frac{1}{2\sqrt{64}}. \]
which gets us the required result.

\[ (h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))} = \frac{1}{h'(0)} = \frac{1}{\sin^2(\sin(1))}. \]

**g)** Since \((f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}\), we can use \((\frac{1}{g(x)})' = -\frac{g'(x)}{[g(x)]^2}\), where \(g(x) = f'(f^{-1}(x))\), and thus \(g'(x) = f''(f^{-1}(x))(f^{-1})'(x)\) to obtain

\[ (f^{-1})''(x) = \frac{-f''(f^{-1}(x))(f^{-1})'(x)}{[f'(f^{-1}(x))]^2} = \frac{-f''(f^{-1}(x))}{[f'(f^{-1}(x))]^3}. \]

**h)** First note that if \(a < x < b\), where \(a\) and \(b\) are any two points in an interval \(I\), then in order to have \(0 < t < 1\), we must have \(t = \frac{x-a}{b-a}\) and \(x = tb + (1-t)a\). So, starting from the definition of convexity seen in class:

\[ \frac{f(x) - f(a)}{x-a} < \frac{f(b) - f(a)}{b-a}, \]

we have

\[ f(x) - f(a) < \frac{x-a}{b-a}(f(b) - f(a)) \]

which becomes

\[ f(tb + (1-t)a) - f(a) < t(f(b) - f(a)) \]

and then

\[ f(tb + (1-t)a) < tf(b) + (1-t)f(a). \]

No a priori information about \(a\) and \(b\) was used in this development, so this works for all points in the interval \(I\).

**i)** 1) \(T_{3,0}(x) = \frac{e}{3} + ex + ex^2 + \left(\frac{5e}{3!}\right)x^3\). \hspace{1cm} 2) \(T_{4,0}(x) = x + x^3\)

**j)** 1) \(c_k = a_k + b_k\). \hspace{1cm} 2) \(c_k = (k+1)a_k\).