MAT 425: Introduction to Analysis I
Assignment #3 Solutions

Set 7 (a)

(i) sup = 1, yes. inf = 0, no.
(ii) sup = 1, yes. inf = −1, yes.
(iii) sup = 1, yes. inf = 0, yes.
(iv) sup = \sqrt{2}, no. inf = 0, yes
(v) sup = +\infty, no. inf = −\infty, no.
(vi) None (emptyset).
(vii) sup = −1/2, no. inf = −\infty, no.
(viii) sup = 3/2, yes. inf = −1, no.

Set 7 (b)

(i) sup = 2; inf = none; \lim = −\infty; \lim = −\infty.
(ii) sup = 1; inf = −1; \lim = 1; \lim = −1.
(iii) sup = 1/2; inf = −1/3; \lim = 0; \lim = 0.
(iv) sup = 2/3; inf = 0.6; \lim = 2/3; \lim = 2/3.
(v) sup = none; inf = none; \lim = +\infty; \lim = −\infty.

Set 7 (c)

(i) Since \(A\) is non-empty, there is some \(x \in A\). Then \(−x\) is in \(−A\), so \(−A\) is non-empty.

(ii) Since \(A\) is bounded above, there is some \(y\) s.t. \(y \geq x\) for all \(x \in A\). Then \(−y \leq −x\) for all \(x \in A\), so \(−y \leq z\) for all \(z \in −A\), so \(−A\) is bounded below.

(iii) Let \(\alpha = \sup(−A)\). Then \(\alpha\) is an upper bound for \(−A\), so, reversing the argument just given, \(−\alpha\) is a lower bound for \(A\). Moreover, if \(\beta\) is any lower bound for \(A\), then \(−\beta\) is an upper bound for \(−A\), so \(−\beta \geq \alpha\), so \(\beta \leq −\alpha\). Thus \(−\alpha\) is the greatest lower bound for \(A\).
Set 8 (a) Take $A = (0,1) \cup (1,2) \cup \{11\}$ and $B = (10,11) \cup (11,12) \cup \{1\}$. Then neither $A$ nor $B$ is open, $A \cap B = \emptyset$, and $A \cup B = (0,2) \cup (10,12)$, which is open.

Set 8 (b) To be a perfect set, $B$ should consist only of limit-points, but all points in $B$ except 0 are not limit-points (for e.g. $1/2$ is certainly not a limit-point since you can find a neighborhood around $1/2$ that doesn’t contain any other point of $B$). And $B$ contains its only limit-point, 0, so $\overline{B} = B$.

Set 8 (c) For example, take $A_k = \{1/k\}$, for $k = 1,2,3,...$. Each of the individual $A_k$’s are closed, but their union, $A = \bigcup_{k=1}^{\infty} A_k = \{1,1/2,1/3,1/4,...\}$ is not closed since $A$ does not contain its limit point, 0.

Set 8 (e) Let $x \in [0,1]$. Let $a = \min(x,1-x)$ and if $a = 0$, set $a = 1$ (i.e. $a$ is the least distance to an endpoint). Construct the sequence $\{x_k\}$, where $x_k = x + a/2^k$. Note that $x_k \in [0,1]$ for $k = 1,2,3,...$.

Now, $\forall n \in \mathbb{N}$, $\exists m \in \mathbb{N}$ s.t. $a/2^m < 1/n$. This implies that for $k > m$, $|x - x_k| = |x - (x - a/2^k)| = |a/2^k| = a/2^k < 1/n$. So $x_k \to x$ and $x$ is a limit-point. Since $x$ was chosen arbitrarily, $[0,1]$ consists entirely of limit-points. $[0,1]$ is perfect.

Set 9 (a) If $A$ is compact, it is also closed. We have seen that the union of open sets is open, so it is impossible for that union, i.e. $\bigcup_{k=1}^{n} B_k$, to be equal to $A$.

Set 9 (b) We saw in class that the Cantor set is closed (as the infinite intersection of closed sets). It is also bounded by 0 and 1. So it is compact.

The easy way to show that the Cantor set is non-empty is to say that by the construction rule, 0 will always remain in the Cantor set. So the set is not empty. (this also works, for e.g., for 1/3, 2/3, etc.)

The other way is to use the nested sequence theorem that we saw in class (also Thm 3.3.3, p. 106 in the book), which says that a nested sequence of non-empty compact sets (which is how the Cantor set is constructed) has non-empty intersection, and therefore, the Cantor set is non-empty.