Set 12 → Differential Calculus

(15) a) Using the definition of the derivative seen in class, show that if \( f(x) = \frac{1}{x} \), then \( f'(x_0) = -\frac{1}{x_0^2} \), for \( x_0 \neq 0 \).

(15) b) Let \( f \) be a function such that \( |f(x)| \leq x^2 \) for all \( x \). Prove that \( f \) is differentiable at 0. (Note that we must have \( f(0) = 0 \).)

(15) c) Give an example of a function \( f \) for which \( \lim_{x \to \infty} f(x) \) exists, but \( \lim_{x \to \infty} f'(x) \) does not exist.

(10) d) Prove that if \( \frac{a_0}{1} + \frac{a_1}{2} + \ldots + \frac{a_n}{n+1} = 0 \), then \( a_0 + a_1x + \ldots + a_nx^n = 0 \) for some \( x \in [0, 1] \).

(15) e) Prove that \( \frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8} \).

(10) f) Suppose \( h \) is a function such that \( h'(x) = \sin^2(\sin(x + 1)) \) and \( h(0) = 3 \). Find \((h^{-1})'(3)\).

(+5) *g) Find a formula for \((f^{-1})''(x)\).

(+10) *h) Show that \( f \) is convex on an interval if and only if for all \( x \) and \( y \) in the interval we have

\[
f(tx + (1-t)y) < tf(x) + (1-t)f(y), \quad \text{for } 0 < t < 1.
\]

(This is just a restatement of the definition of convexity seen in class, but a useful one, which is actually used more often in more advanced situations.)

(20) i) 1) Find the Taylor polynomial of degree 3 at \( x_0 = 0 \) for \( f(x) = e^{e^x} \). 2) Find the Taylor polynomial of degree 4 at \( x_0 = 0 \) for \( f(x) = x^5 + x^3 + x \).

(+10) *j) Suppose that \( a_k \) and \( b_k \) are the coefficients in the Taylor polynomials at \( x_0 \) of \( f \) and \( g \), respectively. In other words, \( a_k = f^{(k)}(x_0)/k! \) and \( b_k = g^{(k)}(x_0)/k! \). Find the coefficients \( c_k \) of the Taylor polynomials at \( x_0 \) of 1) \( f + g \), and 2) \( f' \), in terms of the \( a_k \)'s and the \( b_k \)'s.