Set 7 → Theory of limits.

(15) a) Find the inf and the sup of the following sets. Also decide when the sup and the inf happens to belong to the set.

(i) \{\frac{1}{n} : n \in \mathbb{N}\}.
(ii) \{\frac{1}{n} : n \in \mathbb{Z} and n \neq 0}\}.
(iii) \{x : x = 0 or x = 1/n for some n \in \mathbb{N}\}.
(iv) \{x : 0 \leq x \leq \sqrt{2} and x \in \mathbb{Q}\}.
(v) \{x : x^2 + x + 1 \geq 0\}.
(vi) \{x : x^2 + x + 1 \lt 0\}.
(vii) \{x : x < 0 and x + 2 + x - 1 < 0\}.
(viii) \{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\}.

(10) b) For the following sequences, give sup, inf and \lim, lim.

(i) (2, 1.9, 1.8, 1.7,... , 2 - (k - 1)/10,...)
(ii) (1, -1, 1, -1,... , (-1)^k,...)
(iii) (1/2, -1/3, 1/4, -1/5,... , (-1)^k-1/(k + 1),...)
(iv) (0.6, 0.66, 0.666,... , 2/3(1 - 1/10^k),...)
(v) (-1, 2, -3, 4, -5,... , (-1)^k,...)

(15) c) Suppose A is a nonempty set that is bounded below. Let \(-A\) denote the set of all \(-x\) for \(x \in A\). Prove that (i) \(-A\) is nonempty; (ii) \(-A\) is bounded above; (iii) \(-\sup(-A) = \inf(A)\).

Set 8 → Open Sets, Closed Sets.

(15) a) Give an example of subsets A and B of \mathbb{R} such that all three of the following conditions hold: (i) Neither A nor B is open; (ii) A ∩ B = ∅; (iii) A ∪ B is open.

(15) b) Show that \(B = \{1/k : k = 1, 2, 3,... \} \cup \{0\}\) is not a perfect set, but that \(\overline{B} = B\).

(10) c) Find an infinite collection of closed sets such that their infinite union is not closed. Explain.

(+10) e) Show that [0, 1] is a perfect set.

Set 9 → Compact Sets.

(10) a) If \(B_1, B_2, ..., B_n\) is a finite open cover of a compact set A, can the union \(\bigcup_{k=1}^{n} B_k\) equal A exactly?

(10) b) Show that (i) the Cantor set is compact and (ii) give two different arguments showing that it is non-empty.