FRACTAL DIMENSION OF RESIDUES SETS WITHIN PASCAL’S TRIANGLE UNDER SQUARE-FREE MODULI

by

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Abstract

The image of certain residues modulo \( m \) embedded within Pascal’s Triangle has fractal properties. Depending on the modulus and residue set, we can determine the box-counting fractal dimension. When the modulus is a prime, \( p \), and the residue set \( R \) is any non-empty subset of \(( \mathbb{Z}/p\mathbb{Z})^*\), then the fractal dimension of Pascal’s Triangle mod \( p \) residues \( R \) is

\[
D(p, R) = 1 + \log_p \left( \frac{p + 1}{2} \right).
\]

If \( R^* = (\mathbb{Z}/p^n\mathbb{Z})^* \), then we have

\[
D(p^n, R^*) = D(p, R)
\]

If the modulus is a square-free integer, \( m \), then

\[
D(m, \mathbb{Z}/m\mathbb{Z} - \{0\}) = D(p_m, R),
\]

where \( p_m \) is the largest prime divisor of \( m \) and \( R \) is any non-empty subset of \((\mathbb{Z}/p_m\mathbb{Z})^*\). We have strong reason to believe that for any non-unit modulus \( m \),

\[
D(m, \mathbb{Z}/m\mathbb{Z} - \{0\}) = D(p_m, R),
\]

and have less strong of a reason to believe that this doesn’t extend for any non-unit modulus \( m \) and any non-empty \( R \subseteq \mathbb{Z}/m\mathbb{Z} - \{0\} \), meaning it seems it is not always true that

\[
D(m, R) = D(p_m).
\]
Dedicated to John N. Guidi.

Check out this pattern; it starts with a 1.
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Introduction

0.1 Fractals and Fractal Dimension

The concept of a fractal was first introduced by Benoît Mandelbrot in the early 1980’s [1]. Since then, the study of fractals and their abundance of applications has taken the scientific community by storm. To name just a few of these applications, fractal analysis has been applied to research in detection of salivary glands diseases [2], the Indian stock market [3], soil pore variability [4], cerebellar activity after brain injuries [5], and breast cancer detection and diagnostics [6] [7].

But what exactly is a fractal? Generally, a fractal is an object that exhibits self-similarity. This means that the object breaks into smaller versions of itself, a pattern which continues forever. This self-repeating pattern gives fractals the property that when “zooming up” on a certain section, the close-up will be an exact copy of the original image. Fractals are constructed following a certain rule; having a starting point and then after infinitely iterating that rule, we arrive at the fractal.

A classic example of a fractal is the Cantor Set, $\mathcal{C}$, shown in Figure 1. The Cantor Set is constructed by taking the closed interval $[0,1]$, then removing the open interval that is the middle third, then with the remaining two parts, removing the open interval middle third of those two segments, and so on and so on. The Cantor Set is the collection of points that remain after this iterative process is repeated an infinite number of times.

![Figure 1: First 7 Iterations of the Cantor Set Construction [8].](image-url)
Another example of a fractal is the Koch Snowflake, the first several iterations are shown in Figure 2. In a complete Koch Snowflake, each of the six arms has three miniature arms, upon which are three more miniatures arms, a pattern which continues indefinitely.

![First 6 Iterations of the Koch Snowflake Construction](image)

Figure 2: First 6 Iterations of the Koch Snowflake Construction [9].

Another paragon fractal, and one that is central to this thesis, is Sierpinski’s Triangle, shown in Figure 3.

![First 5 Iterations of the Construction of Sierpinski’s Triangle](image)

Figure 3: First 5 Iterations of the Construction of Sierpinski’s Triangle [10].

Associated with a fractal is its fractal dimension, a numerical value that usually exceeds the fractal’s topological dimension. There are actually several different types of these so-called fractal dimensions; the box-counting dimension, the information dimension, the correlation dimension, generalized/Renyi dimensions (a collection of dimensions of which the three previous dimensions are special cases), and the Higuchi dimension are a few such fractal dimensions. Strictly speaking, the Hausdorff dimension is the true fractal dimension of a shape, while all the previously mentioned fractal
dimensions are tools to approximate this Hausdorff dimension (and sometimes arrive at the exact value.) As the box-counting dimension is the typical dimension used in the study of fractals [1], it is also the fractal dimension we will use from here on out.

**Definition 1: Fractal Dimension:**

Let $S$ be a compact subset of $X$, where $(X,d)$ is a metric space. Let $N(S,\epsilon)$ be the minimum number of closed balls of radius $\epsilon$ needed to cover $S$. The box-counting dimension of $S$, hereafter referred to as the fractal dimension of $S$, is defined as

$$D = \lim_{\epsilon \to 0} \frac{\log(N(S,\epsilon))}{\log(1/\epsilon)}$$

if such a limit exists.

As the name suggests, this should have something to do with counting boxes. That is because of The Box Counting Theorem, which states that the above definition of fractal dimension is exactly equivalent to

$$D = \lim_{k \to \infty} \frac{\log(N(S,\epsilon(k)))}{\log(1/\epsilon(k))},$$

where $\epsilon(k)$ is some positive function of an discrete index $k$, where $\lim_{k \to \infty} \epsilon(k) = 0$, and $N(S,\epsilon(k))$ is the minimum number of squares of side length $\epsilon(k)$ needed to cover $S$ [11].

A simple and rudimentary example is the calculation of the fractal dimension for a solid square. For each time we shrink the radius by a power of 2, the number of boxes it takes to cover the square grows by a multiple of 4 (see Figure 4).

Thus,

$$D = \lim_{\epsilon \to 0} \frac{\log(N(S,\epsilon))}{\log(1/\epsilon)} = \lim_{k \to \infty} \frac{\log(N(S,\epsilon(2^k)))}{\log(2^k)}$$
Figure 4: First several iterations of $\epsilon(k)$ and $N(k)$ for a solid square.

$$= \lim_{k \to \infty} \frac{\log(4^k)}{\log(2^k)} = \lim_{k \to \infty} \frac{k \log(4)}{k \log(2)} = \frac{\log(4)}{\log(2)} = 2,$$

giving us that the fractal dimension of a simple solid square is 2, which is the same value as its topological dimension. But the true purpose of the fractal dimension is to identify fractals, indicated by a non-integer fractal dimension.

Two examples of such fractals, whose $D$-values are easy to calculate, are the aforementioned Cantor Set (Figure 1) and Sierpinski’s Triangle (Figure 3). For the Cantor set, notice that for any one iteration, if we take the ball size to be exactly large enough to cover one of the remaining intervals, then the ball radius is a third of what it was in the previous iteration, while the number of balls required is exactly twice what was required in the previous iteration. More formally,

$$D = \lim_{\epsilon \to 0} \frac{\log(N(C, \epsilon))}{\log(1/\epsilon)} = \lim_{k \to \infty} \frac{\log(N(C, \frac{1}{3k}))}{\log(3^k)}$$

$$= \lim_{k \to \infty} \frac{\log(2^k)}{\log(3^k)} = \lim_{k \to \infty} \frac{k \log(2)}{k \log(3)} = \frac{\log(2)}{\log(3)},$$

Thus, we find that the Cantor Set has a fractal dimension of $\frac{\log(2)}{\log(3)}$. By a similar argument for Sierpinski's Triangle, we see that for each iteration of the triangle, if we take the box size to be exactly large enough to cover one of the remaining solid triangles, then the box radius is one half of what it was in the previous iteration,
while the number of boxes required is exactly three times what was required in the previous iteration (see Figure 5).

![Figure 5: First several iterations of $\epsilon(k)$ and $N(k)$ for Sierpinski’s Triangle.](image)

Again, more formally,

$$D = \lim_{\epsilon \to 0} \frac{\log(N(S, \epsilon))}{\log(1/\epsilon)} = \lim_{k \to \infty} \frac{\log(N(S, \frac{1}{2^k}))}{\log(2^k)}$$

$$= \lim_{k \to \infty} \frac{\log(3^k)}{\log(2^k)} = \lim_{k \to \infty} k \frac{\log(3)}{\log(2)} = \frac{\log(3)}{\log(2)},$$

giving us that Sierpinski’s Triangle as a fractal dimension of $\log(3)/\log(2)$.

From the fractal dimension of a set, $S$, we can determine certain properties about $S$. If $D \notin \mathbb{N}$, then the shape is fractal. The closer $D$ is to a natural number, the “closer” that shape is to being in that Euclidean dimension. For example, Sierpinski’s Triangle has a smaller fractal dimension ($D \approx 1.58$) than the Von Koch curve 85° ($D \approx 1.78$), and notice in Figure (6) that the Von Koch curve 85° is closer to being a 2D solid than Sierpinski’s Triangle.
0.2 Pascal’s Triangle

A seemingly disjoint topic in mathematics is *Pascal’s Triangle* (Figure 7), whose history is not nearly as modern as fractals. In the western world, this triangle is named after 17th century mathematician Blaise Pascal, but was studied by Chinese, Hindu, Arabic, and Greek mathematicians, with an illustration of the figure appearing in a Chinese encyclopedia written in 1407 [13].

In order to explain what exactly Pascal’s Triangle is, it is first required to know the definition of *binomial coefficients*. The number of ways to choose $k$ elements out
of \( n \) elements is the binomial coefficient \( n \) choose \( k \), and is denoted by \( \binom{n}{k} \). From a counting argument, we can see that

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

where \( x! = x \times (x-1) \times (x-2) \times \cdots \times 1 \) for \( x \in \mathbb{Z}^{>0} \), \( 0! = 1 \), and \( x! \) undefined for \( x \in \mathbb{Z}^{<0} \). One can see that for each \( n \), there are \((n+1)\)-different \( k \)'s such that \( \binom{n}{k} \) is defined. Thus, we can arrange the binomial coefficients into a triangle, where \( n \) is the index of a row in the triangle, and \( k \) is an entry in that row. This triangle is Pascal’s Triangle.

### 0.2.1 Properties and Patterns Within Pascal’s Triangle

The triangle is host to a plethora of intriguing properties and patterns. The first is that the triangle is a cellular automaton, in that any one entry in the triangle is determined by other entries. The rule here is that one entry is equal to the sum of the two directly above it, which follows from the property of binomial coefficients:

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n-1)!(k+n-k)}{k!(n-k)!}
\]

\[
= \frac{(n-1)!k}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!}
\]

\[
= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}
\]

\[
= \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} + \frac{(n-1)!}{k!(n-k-1)!}
\]

\[
= \binom{n-1}{k-1} + \binom{n-1}{k}
\]

7
Following this rule as a recurrence, we arrive at the “hockey stick rule” (see Figure 8), that starting on a 1 and adding entries sequentially in a diagonal, then changing direction, the next entry in the new direction is the sum of the previous entries.

The diagonals of the triangle are significant too. In the first diagonal, all entries are 1’s, the second diagonal is the natural numbers in order, the third diagonal is the triangular numbers in order, the next diagonal is the ordered tetrahedral numbers, then the pentatope numbers. In general, the $i^{th}$ term in the $n^{th}$ diagonal is the number of points it takes to construct an $n$-dimensional triangle with length $i$ (here, a line of length $i$ requires $i + 1$ collinear points) [17].

Another pattern found in Pascal’s Triangle is the occurrence of the Fibonacci Numbers. If the rows of the triangle are left-justified, the diagonals and “half-diagonals” sum up to the Fibonacci Numbers, in order (see Figure 9).
Another phenomenal property, and the genesis of this thesis, is the fractal patterns of Pascal's Triangle. If one were to take each entry of Pascal's Triangle, and erase the entry if even, and replace it with a solid square if odd, then the pattern resembles Sierpinski's Triangle when we observe the first $2^k$ rows ($k \in \mathbb{Z}^\geq 0$). Figure 10 shows the first 32 rows of Pascal's Triangle, the odd entries denoted by a 1 (their remainder upon division by 2).

In fact, the larger $k$ is, the closer the pattern resembles Sierpinski's Triangle. Figure 11 shows the first 2048 rows of Pascal's Triangle, odd entries replaced by a black pixel, even entries by a white pixel.

In truth, this process of removing the even entries is a specific case of a more general process.

**Definition 2: Pascal’s Triangle mod $m$ :**

Let us consider the first $m^k$ rows of Pascal’s Triangle, scaled to give the entire triangle...
Figure 10: The first 32 rows of Pascal’s Triangle, with entries written as their mod 2 congruence.

Figure 11: The first 2048 rows of Pascal’s Triangle, even entries represented by a white pixel, odd represented by a white pixel.
a length of 1, and each cell entry a length of $1/m^k$, $m \in \mathbb{Z}^>0$. We replace each entry with its congruence mod $m$. Then, if a cell entry’s remainder mod $m$ is zero, represent it with a white square; if the remainder is non-zero, represent it with a solid black square. When we let $k \to \infty$, the resulting image is referred to as Pascal’s Triangle mod $m$, denoted by $P_m$.

Figure 12: Examples of several $P_m$, cut off at a particular row to illustrate the pattern.

As one can see from a few values of $m$ (Figure 12), if $m$ is a prime, $p$, then a fractal pattern appears: one which is a sort of generalized Sierpinski’s Triangle (where the pattern iterates every $p^k$ rows, $k \in \mathbb{Z}^>0$.) Thus, from here onward, the “$k^{th}$ iteration” of $P_p$ is the first $p^k$ rows of $P_p$.

Also note that if $m$ is of the form $p^n$, $n \in \mathbb{Z}^>0$, then another fractal pattern appears which resembles the mod $p^{n-1}$ triangle, but with copies of the mod $p$ triangle in previous holes.

Finally, if the modulus $m$ has a prime factorization

$$m = \prod_{i=1}^{s} p_i^{k_i}$$
where \( p_i \) is prime, \( k_i \) is the power of \( p_i \) that divides \( m \), and \( s \) is the index such that \( p_s \) is the largest prime dividing \( m \), then

\[
P_m = \bigcup_{i=1}^{s} P_{p_i^{k_i}}
\]  

(1)

In other words, \( P_m \) is just the superposition of all the images of \( m \)'s \( p_i^{k_i} \) divisors. This property is expected. This is because

\[
a \equiv 0 \mod bc \iff (a \equiv 0 \mod b \text{ and } a \equiv 0 \mod c)
\]

\( a, b, c \in \mathbb{Z}, \gcd(b, c) = 1 \). So in order for an entry to be removed mod \( bc \), it has to be removed in both mod \( b \) and mod \( c \). The logically equivalent statement,

\[
a \not\equiv 0 \mod bc \iff (a \not\equiv 0 \mod b \text{ or } a \not\equiv 0 \mod c)
\]

tells us that in order for an entry to not be erased mod \( bc \), it must not be erased mod \( b \) or it must not be erased in mod \( c \). These two statements together tell us that the image \( P_{bc} \) is the superposition of the images \( P_b \) and \( P_c \), when \( b \) and \( c \) are relatively prime. Because all the \( p_i^{k_i} \) divisors of \( m \) are relatively prime, (1) follows.

Even further beyond these patterns, we can observe a more impressive phenomenon. But first we must define a new term.

**Definition 3: Pascal's Triangle mod m residue r:**

Let us take the first \( m^k \) rows of Pascal's Triangle, rescaled to have the whole triangle have a length of 1, and replace each entry with its congruence mod \( m \), \( m \in \mathbb{Z}^>0 \). Then, if a cell entry's remainder mod \( m \) is not \( r \), represent it with a white square; if the remainder is \( r \), represent it with a solid black square. When we let \( k \to \infty \), the
resulting image is referred to as *Pascal’s Triangle mod m residue r*, denoted by $P_{m,r}$. Note that $\bigcup_{r=1}^{m-1} P_{m,r} = P_m$.

By viewing a few examples (see Figure 13), we can see that $P_{p,r}$ strongly resembles $P_p$, for any $r$.

![Figure 13: Examples of Several $P_{p,r}$](image1.png)

Further, we can see in Figure 14 that for a fixed $p$, the various $P_{p,r}$ more closely resemble $P_p$ as the number of rows we view increases.

![Figure 14: Illustration of how various $P_{p,r}$ appear to “converge” to $P_p$ as the number of rows used to display the image is increased.](image2.png)

### 0.2.3 Fractal Dimension of Pascal’s Triangle mod $m$

If we can see that the triangles $P_m$ appear to be fractal in nature, a logical question to ask would be what the dimension of $P_m$ is, as a function of $m$, denoted by $D(m)$.
It is already known [18] that for \( p \) prime, the fractal dimension of \( P_p \) is

\[
D(p) = 1 + \log_p \left( \frac{p + 1}{2} \right).
\]

However, it is still unknown what \( D(m) \) for a general \( m \) is. It will be the endeavor of this thesis to extend the previous statement to square-free \( m \), but also to show that for \( p \) prime, the fractal dimension of \( P_p,r \), denoted by \( D(p, \{r\}) \) is also

\[
D(p, \{r\}) = 1 + \log_p \left( \frac{p + 1}{2} \right).
\]

That is, as visually hinted at in the previous section, the fractal dimension of \( P_{p,r} \) does not depend on \( r \). To take this a step further, let us define a new term, \( P_{p,R} \):

**Definition 4: Pascal’s Triangle mod p residues \( R \):**

Let us take the first \( p^k \) rows of Pascal’s Triangle, rescaled to have the whole triangle have a length of 1, and replace each entry with its congruence mod \( p \), \( p \) prime. Let \( R \subseteq (\mathbb{Z}/p\mathbb{Z})^*, R \neq \emptyset \). Then, if a cell entry’s remainder mod \( p \) is not an element of \( R \), represent it with a white square; if the remainder is an element of \( R \), represent it with a solid square. When we let \( k \to \infty \), the resulting image is referred to as *Pascal’s Triangle mod p residues \( R \)*, denoted by \( P_{p,R} \).

This thesis will prove that the fractal dimension of \( P_{p,R} \), denoted by \( D(p, R) \) is also

\[
D(p, R) = 1 + \log_p \left( \frac{p + 1}{2} \right).
\]

That is, not only the image of any singleton residue, but the image of any non-empty combination of non-zero residues in \( P_p \) will have the same fractal dimension.

We will also extend this to two more statements, both of which are more general
but in different ways. First, if $R = \mathbb{Z}/m\mathbb{Z} - \{0\}$, then $D(m, R)$, written as $D(m)$, is

$$D(m) = 1 + \log_{p_m} \left( \frac{p_m + 1}{2} \right)$$

where $m$ is a square-free integer, and $p_m$ is the largest prime divisor of $m$.

The other general statement is that when $R = R^* = (\mathbb{Z}/p^n\mathbb{Z})^*$, $n \in \mathbb{Z}^{>0}$, then we have

$$D(p^n, R^*) = 1 + \log_p \left( \frac{p + 1}{2} \right).$$
1 Method For Calculating $D$

Our first task is to establish a way to calculate $D$. Our method will involve counting cell entries, and we will first use it to confirm $D(p) = 1 + \log_p \left( \frac{p+1}{2} \right)$. First, let

$$N_k(p, R) = \sum_{r \in R} \text{number of } r \text{-cells in first } p^k \text{ rows of } P_p,$$

where $p$ is prime, $\emptyset \neq R \subseteq (\mathbb{Z}/p\mathbb{Z})^*$, and an $r$-cell is a cell-entry in $P_p$ which is $r \mod p$. For shorthand, if $R = (\mathbb{Z}/p\mathbb{Z})^*$, then we say that $N_k(p, R) = N_k(p)$.

As we saw in examples in Section 0.1, we can think of $D$ as being

$$D = \lim_{k \to \infty} \frac{\log(N(S, \epsilon(k)))}{\log(1/\epsilon(k))}.$$

This expression and (2) work together quite conveniently. If we consider iterations of $P_p$ by looking at the first $p^k$ rows, and let our box be the size of one cell entry, then $\epsilon(k)$ is $\frac{1}{p^k}$ and $N(P_p, \epsilon(k))$ is $N_k(p, R)$ (see Figure 15).

Putting this all together, we end up with

$$D(p, R) = \lim_{k \to \infty} \frac{\log(N_k(p, R))}{\log(p^k)} \quad (3)$$

which we can quickly show agrees with the known relationship

$$D(p) = 1 + \log_p \left( \frac{p + 1}{2} \right).$$

For a given prime, $p$, the entire growth of Pascal’s Triangle mod $p$ is determined by its growth triangle, $T_p$. In one iteration of $P_p$, a cell-entry $e$ corresponds to a
Figure 15: $N_k(p, R)$ entries of size $\frac{1}{p^k}$ (here $p = 2$ and $R = \{1\}$).

triangle of cell-entries in the next iteration, where the triangle is simply $e \cdot T_p$ (each entry of $T_p$ is multiplied by $e \mod p$.) See Figure 16. The proof of this is a lengthy endeavor of its own, and fortunately has already been accomplished. For further information, read “Granville’s Zaphod Beeblebrox’s Brain and the Fifty-ninth Row of Pascal’s Triangle” [19].

It is easy to see that for each $p$, $T_p$ is simply the first $p$ rows of $P_p$. This is because the $0^{th}$ iteration of $P_p$’s pattern is the first $p^0 = 1$ row, which is always the singleton entry 1. By the property of $T_p$ which dictates that a 1 corresponds to $1 \cdot T_p$ in the next iteration, we see that the triangle present in the first $p^1$ rows must be $T_p$.

It is a quick consequence that $D(p) = 1 + \log_p \left( \frac{p+1}{2} \right)$. Because each entry in one iteration corresponds to a triangle in the next iteration, and that triangle has $\frac{p(p+1)}{2}$
Figure 16: Example of $T_p$ role in determining the pattern of $P_p$. Here, $p=5$.

entries (because it is the first $p$ rows), we can see that $N_{k+1}(p) = \frac{p(p+1)}{2} N_k(p)$ and $N_0(p) = 1$, which implies $N_k(p) = \left(\frac{p(p+1)}{2}\right)^k$. Using this with (3) gives us

$$D(p) = \lim_{k \to \infty} \frac{\log \left( \frac{p(p+1)^k}{2^k} \right)}{\log(p)} = \lim_{k \to \infty} \frac{k \log \left( \frac{p(p+1)^k}{2^k} \right)}{k \log(p)} = \log \left( \frac{p(p+1)^k}{2^k} \right) = 1 + \log_p \left( \frac{p+1}{2} \right),$$

which is in agreement with the known expression for $D(p)$, but determined in a different way [18].

An interesting and seemingly irrelevant fact can be observed from the existence of $T_p$, a fact which as it were, turns out to be essential later in this thesis.
**Lemma 1.** For any two residues, \( r_1, r_2 \in (\mathbb{Z}/p\mathbb{Z})^* \), \( p \) prime, the ratio of their occurrences in \( P_p \),

\[
F_{r_1,r_2} = \lim_{k \to \infty} \frac{N_k(p, \{r_1\})}{N_k(p, \{r_2\})},
\]
does not converge to zero.

**Proof:** Let the number of times the residue \( r \) appears in \( e \cdot T_p \) be denoted by \( C_e(r) \), \((e \in (\mathbb{Z}/p\mathbb{Z})^*)\).

For any \( r \) and \( e \), \( r \) appears in \( e \cdot T_p \) a known number of times. More specifically, \( C_e(r) = C_1(r^*) \), where \( r^* \in (\mathbb{Z}/p\mathbb{Z})^* \) s.t. \( r^* \cdot e \equiv r \mod p \). In other words, there will be a certain number of \( r^* \) in \( T_p \) that when multiplied by \( e \) will be congruent to \( r \mod p \), and thus that many \( r \) will be in \( e \cdot T_p \).

We know that \( T_p \) contains every residue in \( (\mathbb{Z}/p\mathbb{Z})^* \) (recall, the second diagonal of Pascal’s Triangle is the natural numbers). Thus, we know that \( r^* \) occurs at least once. Therefore, \( C_e(r) = C_1(r^*) \geq 1 \). Also note that \( C_e(r) \leq \frac{p(p+1)}{2} - p + 2 \), because \( e \cdot T_p \) has only \( \frac{p(p+1)}{2} \) entries, and there are \( p - 2 \) other residues that must occur at least once.

Now, suppose we want to find the ratio of occurrences of one residue, \( r_1 \), compared to the occurrences of another residue, \( r_2 \). We name this value \( F_{r_1,r_2} \), and is evaluated as

\[
F_{r_1,r_2} = \lim_{k \to \infty} \frac{N_k(p, \{r_1\})}{N_k(p, \{r_2\})}.
\]
Now, at every iteration of \( k \), we know that \( P_p \) is composed of copies of some variety of \( T_p, 2 \cdot T_p, \ldots, (p-1) \cdot T_p \) (one triangle for each non-zero entry in the previous
iteration of \( k \). Thus we can rewrite \( F_{n_1, n_2} \) as

\[
F_{n_1, n_2} = \frac{\sum_{e T_p \text{ in } P_p} C_e(r_1)}{\sum_{e T_p \text{ in } P_p} C_e(r_2)},
\]

which, from the bounds we found above, can confirm is at its smallest

\[
F_{n_1, n_2} \geq \frac{\sum_{e T_p \text{ in } P_p} 1}{\sum_{e T_p \text{ in } P_p} \frac{p(p+1)}{2} - p + 2} = \frac{1}{\frac{p(p+1)}{2} - p + 2}
\]

and at its largest

\[
F_{n_1, n_2} \leq \frac{\sum_{e T_p \text{ in } P_p} \frac{p(p+1)}{2} - p + 2}{\sum_{e T_p \text{ in } P_p} 1} = \frac{\frac{p(p+1)}{2} - p + 2}{1} = \frac{p(p+1)}{2} - p + 2.
\]

Thus,

\[
\frac{1}{\frac{p(p+1)}{2} - p + 2} \leq F_{n_1, n_2} \leq \frac{p(p+1)}{2} - p + 2,
\]

if \( F_{n_1, n_2} \) exists. This shows that \( F_{n_1, n_2} \) is never zero, for any combination of residues.

\[\blacksquare\]
2 Arriving At An Expression For $N_k(p, R)$

Before we can use (3) to find the fractal dimension of $P_{p,R}$, we still need to find $N_k(p, R)$ for a general non-empty $R \subseteq (Z/pZ)^*$. Note that

$$N_k(p, R) = \sum_{r \in R} N_k(p, \{r\}).$$

(4)

Thus, a logical place to begin would be to find $N_k(p, \{r\})$ first.

2.1 Rudimentary Expression for $N_k(p, \{r\})$

Recall, for a fixed $r \in (Z/pZ)^*$, some of the $r$-cells in one iteration of $P_p$ are generated by a 1 in the previous iteration, others are determined by a previous 2, etc. For a particular entry $e$ of the $k^{th}$ iteration of $P_p$, the number of $r$’s it generates in the $(k + 1)^{th}$ iteration of $P_p$ is equal to the number of $r$’s in $e \cdot T_p$. It follows that for a fixed $r$, the number of $r$-cells in current iteration = the number of $r$’s in $1 \cdot T_p$ \cdot (number of 1’s in previous iteration) + the number of $r$’s in $2 \cdot T_p$ \cdot (number of 2’s in previous iteration) + . . . + the number of $r$’s in $(p - 1) \cdot T_p$ \cdot (number of $(p - 1)$’s in previous iteration). Or, more precisely, for a given $r$,

$$N_k(p, \{r\}) = \sum_{l=1}^{p-1} (\# \ of \ r\text{-}cells\ in \ l \cdot T_p)N_{k-1}(p, \{l\}).$$

(5)

From this, we let
M_p = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,p-1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,p-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{p-1,1} & m_{p-1,2} & \cdots & m_{p-1,p-1} \end{bmatrix} \quad \text{and} \quad \vec{v}_k = \begin{bmatrix} N_k(p, \{1\}) \\ \vdots \\ N_k(p, \{r\}) \\ \vdots \\ N_k(p, \{p-1\}) \end{bmatrix}

where m_{i,j} = \text{number of } i\text{-cells in } j \cdot T_p. \text{ Then we have }

\vec{v}_k = M_p \vec{v}_{k-1}.

A direct consequence of this is that

\vec{v}_k = M_p^k \vec{v}_0 \quad \forall \; k \in \mathbb{N}.

\textbf{Theorem 2.} For a given prime } p \text{ and residue } r, \text{ the number of cells of residue } r\text{ in the first } p^k \text{ rows of } P_p \text{ is }

N_k(p, \{r\}) = \sum_{\lambda_i} |\lambda_i|^k q_i(k)[B_i \cos(\theta_i k) + C_i \sin(\theta_i k)],

where each } \lambda_i \text{ is an eigenvalue of } M_p, \text{ } q_i(k) \text{ is a polynomial in } k, \text{ whose degree is } l_i - 1, \text{ where } l_i \text{ is } \lambda_i\text{'s multiplicity as a root of } M_p\text{'s characteristic polynomial, } \theta_i \text{ is the argument of } \lambda_i, \text{ and } B_i \text{ and } C_i \text{ are constants. The explicit values of all constant in this expression are determined by the chosen } r.

\textit{Proof:} Let } M_p\text{'s characteristic polynomial be }

\beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_{p-1} x^{p-1} + x^p = f(x).
By the Cayley-Hamilton Theorem, \( f(M_p) = 0_{(p-1,p-1)} \) (matrix of zeros), which allows us to make the following manipulations:

\[
\beta_0 I + \beta_1 M_p + \beta_2 M_p^2 + \cdots + \beta_{p-1} M_p^{p-1} + M_p^p = 0
\]

\[
M_p^p = -\beta_{p-1} M_p^{p-1} - \cdots - \beta_1 M_p - \beta_0 I
\]

Multiply by \( M_p^{k-p} \), for some arbitrary \( k \in \mathbb{N}, k \geq p \)

\[
M_p^p M_p^{k-p} = -\beta_{p-1} M_p^{p-1} M_p^{k-p} - \cdots - \beta_1 M_p^{k-p} - \beta_0 I M_p^{k-p}
\]

\[
M_p^k = -\beta_{p-1} M_p^{k-1} - \cdots - \beta_1 M_p^{(p-1)} - \beta_0 M_p^{k-p}
\]

Multiplying by \( \vec{v}_0 \),

\[
M_p^k \vec{v}_0 = \vec{v}_k = -\beta_{p-1} M_p^{k-1} \vec{v}_0 - \cdots - \beta_1 M_p^{(p-1)} \vec{v}_0 - \beta_0 M_p^{k-p} \vec{v}_0
\]

and because of (7),

\[
\vec{v}_k = -\beta_{p-1} \vec{v}_{k-1} - \cdots - \beta_1 \vec{v}_{(p-1)} - \beta_0 \vec{v}_{k-p},
\]

which is true \( \forall k \in \mathbb{N}, k \geq p \). This is a recurrence relation for \( \vec{v}_k \). Recall that \( \vec{v}_k \) is a vector of recurrence terms, meaning this general recurrence holds for each \( N_k(p, \{ r \}) \) entry in \( \vec{v}_k \), regardless of the \( r \). Therefore, for each \( r \),

\[
N_k(p, \{ r \}) = -\beta_{p-1} N_{k-1}(p, \{ r \}) - \beta_{p-2} N_{k-2}(p, \{ r \}) - \cdots - \beta_0 N_{k-p}(p, \{ r \}).
\]

To find the explicit equation for this linear homogeneous recurrence relation with
constant coefficients, we need to find each root, $\lambda_i$, of the polynomial

$$x^p + \beta_{p-1}x^{p-1} + \cdots + \beta_0 = f(x),$$

which of course is the characteristic polynomial of $M_p$, with roots equal to the eigenvalues. Once we know the roots, we can make the rather general statement

$$N_k(p, \{r\}) = \text{the sum of several terms},$$

where the terms are wholly determined by the collection of eigenvalues. Namely, for each eigenvalue, $\lambda_i$, either

- $\lambda_i$ is real, in which case

  $$A_0\lambda_i^k + A_1 k\lambda_i^k + \cdots + A_{l_i-1} k^{l_i-1} \lambda_i^k = \lambda_i^k \sum_{n=1}^{l_i-1} A_n k^n = \lambda_i^k q_i(k)$$

  appears as a term in $N_k(p, \{r\})$, where each $A_n$ is a constant, $l_i$ is the multiplicity of $\lambda_i$ as a root, and $q_i(k)$ is an $(l_i - 1)$-degree polynomial in $k$ (but depending on the coefficients could be identically zero).

- $\lambda_i$ is non-real, in which case $\lambda_i$ and $\overline{\lambda_i}$ are both roots, and

  $$|\lambda_i|^k [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)]$$

  appears as a term in $N_k(p, \{r\})$, where $B_i$ and $C_i$ are constants (potentially zero), and $\theta_i$ is the argument of $\lambda_i$ with $0 \leq \theta_i < 2\pi$.
Therefore,

\[ N_k(p, \{ r \}) = \sum_{\lambda_i \in \mathbb{R}} \lambda_i^k q_i(k) + \sum_{\lambda_j \in \mathbb{C} - \mathbb{R}} |\lambda_j|^k [B_j \cos(\theta_j k) + C_j \sin(\theta_j k)]. \]

Or, alternatively, we can say

\[ N_k(p, \{ r \}) = \sum_{\lambda_i} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)] \] (8)

where \( B_i = 1 \) if \( \lambda_i \in \mathbb{R} \), and \( q_i(k) = 1 \) if \( \lambda_i \in \mathbb{C} - \mathbb{R} \).

\[ \blacksquare \]

### 2.2 Refined Expression for \( N_k(p, \{ r \}) \)

**Theorem 3.** The eigenvalue of \( M_p \) with the largest magnitude is \( \frac{p(p+1)}{2} \).

Granville [19] showed that \( T_p \) consists of \( \frac{p(p+1)}{2} \) entries, none of which are zero mod \( p \). The fact that each entry is non-zero mod \( p \) implies that the set of entries in \( T_p \) is \((\mathbb{Z}/p\mathbb{Z})^* = G\).

Now recall that

\[
M_p = \begin{bmatrix}
m_{1,1} & m_{1,2} & \cdots & m_{1,p-1} \\
m_{2,1} & m_{2,2} & \cdots & m_{2,p-1} \\
\vdots & \vdots & \ddots & \vdots \\
m_{p-1,1} & m_{p-1,2} & \cdots & m_{p-1,p-1}
\end{bmatrix},
\]

where \( m_{i,j} = \) number of \( i \)-cells in \( j \cdot T_p \),

\[ \Rightarrow m_{i,1} = \) number of \( i \)-cells in \( 1 \cdot T_p \). In other words, the column \( m_{i,1} \) is just the counts
for the number of times an element $g \in G$ appears in $T_p$, where each $i$ corresponds to an individual $g$. Perhaps a redundant restatement, but it is important to point out that for each entry $m_{i,1}$

$$m_{i,1} = \text{the number of } g's \text{ in } T_p \text{ s.t. } g \equiv i \mod p.$$  

(9)

In the process of proving our hypothesis, we shall show that for a fixed $i$, the row $m_{i,j} (j = 1, \ldots, p - 1)$ is a permutation of the column $m_{i,1}$.

Let us generalize (9). Recall $m_{i,j} =$ number of $i$-cells in $j \cdot T_p$. In other words, when $T_p$ is multiplied by $j$, there will be $m_{i,j}$-many $i$’s. Or,

$$m_{i,j} = \text{the number of } g \in G \text{ in } T_p \text{ s.t. } j \cdot g \equiv i \mod p.$$  

(10)

Now let us fix $i$. Consider $m_{i,j}$. Because $0 < j < p$, then $j \in G$, and therefore $j^{-1}$ exists. So therefore

$$m_{i,j} = \text{number of } g \in G \text{ in } T_p \text{ s.t. } j \cdot g \equiv i \mod p$$

$$= \text{number of } g \in G \text{ in } T_p \text{ s.t. } g \equiv j^{-1} \cdot i \mod p = m_{(j^{-1} \cdot i),1} \text{ by (9)}.$$  

Therefore $m_{i,j} = m_{(j^{-1} \cdot i),1}$

Now because we fixed $i$, we see that each $j$ corresponds to a unique $j^{-1} \cdot i$, thus each $m_{i,j}$ in this row corresponds to a unique entry in the column $m_{i,1}$. Thus this row is a permutation of the column $m_{i,1}$.
To illustrate this, take for example $M_7$:

$$M_7 = \begin{bmatrix}
15 & 2 & 2 & 1 & 4 & 4 \\
1 & 15 & 4 & 2 & 4 & 2 \\
4 & 2 & 15 & 4 & 1 & 2 \\
2 & 1 & 4 & 15 & 2 & 4 \\
2 & 4 & 2 & 4 & 15 & 1 \\
4 & 4 & 1 & 2 & 2 & 15
\end{bmatrix},$$

which was constructed using the fact that $m_{i,j} =$ number of $i$-cells in $j \cdot T_7$. Let us check a few $m_{i,j}$.

$m_{1,2} = 2$. Now $j^{-1} \cdot i \equiv 2^{-1} \cdot 1 \equiv 4 \cdot 1 \equiv 4 \text{ mod } 7$. Now $m_{4,1} = 2$, as predicted.

$m_{4,6} = 4$. Now $j^{-1} \cdot i \equiv 6^{-1} \cdot 4 \equiv 6 \cdot 4 \equiv 3 \text{ mod } 7$. Now $m_{3,1} = 4$, as predicted.

$m_{5,5} = 15$. Now $j^{-1} \cdot i \equiv 5^{-1} \cdot 5 \equiv 3 \cdot 5 \equiv 1 \text{ mod } 7$. Now $m_{1,1} = 15$, as predicted.

Now recall that $T_p$ consists of $\frac{p(p+1)}{2}$ entries, and column $m_{i,1}$ is just the counts of particular entries. Thus, the sum of the entries in column $m_{i,1}$ is $\frac{p(p+1)}{2}$. Because each row in $M_p$ is a permutation of $m_{i,1}$, the sum of each row is also $\frac{p(p+1)}{2}$. In other words,

$$M_p \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix} = \begin{bmatrix}
\frac{p(p+1)}{2} \\
\frac{p(p+1)}{2} \\
\vdots \\
\frac{p(p+1)}{2}
\end{bmatrix} = \frac{p(p+1)}{2} \begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix},$$

$\Rightarrow [1, 1, \cdots, 1]^T$ is an eigenvector, with $\frac{p(p+1)}{2}$ as the corresponding eigenvalue. So $\frac{p(p+1)}{2}$ is an eigenvalue of $M_p$. 

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The Perron–Frobenius Theorem states that a square matrix of only positive entries will have only one eigenvector with all positive entries, and that vector’s corresponding eigenvalue is the eigenvalue with the largest magnitude [20]. Our $M_p$ matrix is square, with only positive entries (recall an entry is a count, none of which are zero). Because $\frac{p(p+1)}{2}$’s corresponding eigenvector is all positive, by the Perron–Frobenius Theorem $\frac{p(p+1)}{2}$ is the largest eigenvalue of $M_p$.

Now that we know $\frac{p(p+1)}{2}$ is the largest $|\lambda_i|$ in the expression (8), we can refine our earlier expression for $N_k(p, \{r\})$ to

$$N_k(p, \{r\}) = \left(\frac{p(p+1)}{2}\right)^k q_r(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k)[B_i \cos(\theta_i k) + C_i \sin(\theta_i k)],$$

(11)

where $q_r(k)$ is a polynomial, the degree being one less than the multiplicity of $\frac{p(p+1)}{2}$ as a root of $M_p$’s characteristic polynomial. As written, $q_r(k)$ will depend on $r$, and could possibly be identically zero. However, we shall refine (11) one step further:

**Lemma 4.** $q_r(k)$ is a constant-valued function such that $0 < q_r(k) < 1$.

**Proof** First, recall (4), that

$$N_k(p, R) = \sum_{r \in R} N_k(p, \{r\}).$$

Thus, if we let $R = (\mathbb{Z}/p\mathbb{Z})^*$, then we have

$$\left(\frac{p(p+1)}{2}\right)^k = N_k(p, (\mathbb{Z}/p\mathbb{Z})^*) = \sum_{r \in (\mathbb{Z}/p\mathbb{Z})^*} N_k(p, \{r\}).$$
\[ \sum_{r \in (\mathbb{Z}/p\mathbb{Z})^*} \left(\frac{p(p+1)}{2}\right)^k q_r(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)] \].

But, note that there is no combination of \(\lambda_i\)’s such that

\[ \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)] \]

can equal \(g(k)\left(\frac{p(p+1)}{2}\right)^k\), where \(g(k)\) is some polynomial. This means that in order to have the whole sum equal \(\left(\frac{p(p+1)}{2}\right)^k\), then it must be true that

\[ \left(\frac{p(p+1)}{2}\right)^k = \sum_{r \in (\mathbb{Z}/p\mathbb{Z})^*} \left(\frac{p(p+1)}{2}\right)^k q_r(k), \]

which then implies

\[ \sum_{r \in (\mathbb{Z}/p\mathbb{Z})^*} q_r(k) = 1. \quad (12) \]

Now assume that for any one particular \(q_r(k)\), the leading term has an \(a_n k^n\) term, where \(a_n\) is a positive constant and \(n \in \mathbb{N}\). We will show how this leads to a contradiction.

In order for (12) to hold true, there must be at least one other \(q_r(k)\) with a \(b_n k^n\) term, where \(b_n\) is a negative constant. Consider

\[ N_k(p, r) = \left(\frac{p(p+1)}{2}\right)^k q_r(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)]. \]

Because \(q_r(k)\)’s leading term is negative, and \(\frac{p(p+1)}{2}\) is larger than all \(|\lambda_i|\), it follows
that

\[ \lim_{k \to \infty} N_k(p, r_b) = -\infty, \]

which is impossible, because \( N_k(p, R) \), for any \( R \), is always non-negative (it is just the counts of a particular occurrence.) This tells us each \( q_r(k) \) must have degree 0 (which, as a complete aside, tells us that \( \frac{p(p+1)}{2} \) is the root of \( M_p \) only once.)

We now know each \( q_r(k) \) is just a constant-valued function: positive, negative, or identically zero.

Suppose for a particular \( r_{neg} \), \( q_{r_{neg}}(k) \in \mathbb{R}^<0 \). This would imply

\[ \lim_{k \to \infty} N_k(p, r_{neg}) = -\infty \]

again, which is a contradiction. Therefore, no \( q_r(k) \) can be negative, it must be a non-negative constant (still leaving the possibility that a \( q_r(k) \) could be zero.)

Suppose for a particular residue, \( r_a \), its polynomial \( q_{r_a}(k) = 0 \). There must be at least one residue, \( r_b \), such that its polynomial \( q_{r_b}(k) \neq 0 \). (This must be true because of (12).) Suppose we want to look at the ratio of occurrences of \( r_a \) to the occurrences of \( r_b \). This is then

\[
F_{r_a, r_b} = \lim_{k \to \infty} \frac{N_k(p, \{r_a\})}{N_k(p, \{r_b\})}
\]

\[
= \lim_{k \to \infty} \frac{(\frac{p(p+1)}{2})^k q_{r_a}(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)]}{(\frac{p(p+1)}{2})^k q_{r_b}(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)]}
\]

\[
= \lim_{k \to \infty} \frac{\sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)]}{\sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)]} = 0
\]
because $\frac{p(p+1)}{2}$ is larger than any $|\lambda_i|$. But, recall from Lemma (1), it is impossible for this to be true. Therefore, there cannot be an $r$ such that $q_r(k) = 0$.

Our only remaining option is that each $q_r(k)$ is a positive constant, which must be less than 1 (in order for them to sum to 1.)

\[\blacksquare\]

### 2.3 Expression for $N_k(p, R)$

Finally, we can extend our knowledge of $N_k(p, \{r\})$ to a general expression for $N_k(p, R)$.

**Theorem 5.**

\[
N_k(p, R) = \frac{(p(p+1))}{2} q_R(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta, k) + C_i \sin(\theta, k)],
\]

where $q_R(k)$ is a positive constant, $q_i(k)$ is a polynomial in $k$, whose degree is $l_i - 1$, where $l_i$ is $\lambda_i$’s multiplicity as a root of $M_p$’s characteristic polynomial, $\theta_i$ is the argument of $\lambda_i$, and $B_i$ and $C_i$ are constants. The explicit values of all constant in this expression are determined by the chosen $R$.

Recall from (4) that $N_k(p, R)$ is just the sum of $N_k(p, \{r\})$ terms. Applying this to (11), we get

\[
N_k(p, R) = \sum_{r \in R} N_k(p, \{r\})
= \sum_{r \in R} \left( \left( \frac{p(p+1)}{2} \right)^k q_r(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k) [B_i \cos(\theta, k) + C_i \sin(\theta, k)] \right)
\]

The values of the coefficients of $q_i(k)$ and the constants $B_i$ and $C_i$ are determined by initial conditions of $N_k(p, \{r\})$, which vary depending on the residue. However, each
$r$ has the same $\lambda_i$'s. Suppose for a particular fixed $\lambda$, possibly $\frac{p(p+1)}{2}$, we add two of their terms together. If $\lambda \in \mathbb{R}$, then we have

$$|\lambda|^k q_i(k) + |\lambda|^k q_j(k) = |\lambda|^k q(k)$$

where $q(k)$ is an $(l-1)$-degree polynomial in $k$ (potentially zero if all the coefficients are), where $l$ is multiplicity of $\lambda$ as a root. If $\lambda \in \mathbb{C} - \mathbb{R}$, then we have

$$|\lambda|^k [B_i \cos(\theta k) + C_i \sin(\theta k)] + |\lambda|^k [B_j \cos(\theta k) + C_j \sin(\theta k)]$$

$$= |\lambda|^k [B \cos(\theta k) + C \sin(\theta k)]$$

where $B = B_i + B_j$ and $C = C_i + C_j$ (again, both $B$ and $C$ could be zero). Thus, when adding two or more $N_k(p, \{r\})$ terms, the resulting expression with still be in the form shown in (11). Thus, we arrive at

$$N_k(p, R) = \left( \frac{p(p+1)}{2} \right)^k q_R(k) + \sum_{\lambda_i} |\lambda_i|^k q_i(k) [B_i \cos(\theta_i k) + C_i \sin(\theta_i k)], \quad (13)$$

where $q_R(k)$ is a positive constant, $q_i(k)$ is a polynomial in $k$, whose degree is $l_i - 1$, where $l_i$ is $\lambda_i$'s multiplicity as a root of $M_p$'s characteristic polynomial, $\theta_i$ is the argument of $\lambda_i$, and $B_i$ and $C_i$ are constants.

\[\blacksquare\]
3 Putting The Two Together

Combining the conclusions from Chapters 1 and 2, we have all the tools we need to be able to determine the fractal dimension of certain subsets of residues within Pascal’s Triangle.

3.1 Proving $D(p, R) = 1 + \log_p\left(\frac{p+1}{2}\right)$

Theorem 6.

$$D(p, R) = 1 + \log_p\left(\frac{p+1}{2}\right)$$

for nonempty $R \subseteq (Z/pZ)^*$.

Proof: Combining our expression for the fractal dimension of $P_{p,R}$ (3) and our expression for the numerator of that term (13):

$$D(p, R) = \lim_{k \to \infty} \frac{\log(N_k(p, R))}{\log(p^k)}$$

and

$$N_k(p, R) = \left(\frac{p(p+1)}{2}\right)^k q_R(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k)[B_i \cos(\theta_i k) + C_i \sin(\theta_i k)],$$

we arrive at

$$D(p, R) = \lim_{k \to \infty} \frac{\log\left(\left(\frac{p(p+1)}{2}\right)^k q_R(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k)[B_i \cos(\theta_i k) + C_i \sin(\theta_i k)]\right)}{\log(p^k)}$$

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\[
\lim_{k \to \infty} \log \left( \left( \frac{p(p+1)}{2} \right)^k q_R(k) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k)O(1) \right) / \log(p^k),
\]

where \( g(x) = O(f(x)) \iff \exists C, x_0 \in \mathbb{R} \text{ s.t. } \forall y > x_0, g(y) \leq Cf(y) \). Further, because \( q_R(k) \) is a positive constant (Lemma 4),

\[
\log \left( \left( \frac{p(p+1)}{2} \right)^k q_R(k) \left( 1 + \frac{\sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k)O(1)}{(\frac{p(p+1)}{2})^k q_R(k)} \right) \right) = \lim_{k \to \infty} \frac{\log \left( \left( \frac{p(p+1)}{2} \right)^k q_R(k) \left( 1 + \frac{\sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^k q_i(k)O(1)}{(\frac{p(p+1)}{2})^k q_R(k)} \right) \right)}{\log(p^k)},
\]

and because \( \frac{p(p+1)}{2} \) is larger than all other \(|\lambda_i|\) (because of Theorem 3),

\[
= \lim_{k \to \infty} \frac{\log \left( \left( \frac{p(p+1)}{2} \right)^k q_R(k) \right) + \log(1 + 0)}{\log(p^k)} = \lim_{k \to \infty} \frac{\log \left( \left( \frac{p(p+1)}{2} \right)^k q_R(k) \right) + \log(q_R(k))}{\log(p^k)}
\]

\[
= \lim_{k \to \infty} \frac{\log \left( \left( \frac{p(p+1)}{2} \right)^k \right) + \log(q_R(k))}{\log(p^k)} = \lim_{k \to \infty} \frac{\log \left( \left( \frac{p(p+1)}{2} \right)^k \right)}{\log(p^k)}
\]

\[
= \lim_{k \to \infty} \frac{k \cdot \log \left( \left( \frac{p(p+1)}{2} \right)^k \right)}{k \cdot \log(p^k)} = \frac{\log(p(p+1))}{\log(p)} = 1 + \log_p \left( \frac{p + 1}{2} \right).
\]

Thus, regardless of the set of residues \( R \),

\[
D(p, R) = 1 + \log_p \left( \frac{p + 1}{2} \right).
\]
3.2 Proving $D(p^n, R^*) = 1 + \log_p\left(\frac{p+1}{2}\right)$

**Theorem 7.** For $R^* = (\mathbb{Z}/p^n\mathbb{Z})^*$

$$D(p^n, R^*) = 1 + \log_p\left(\frac{p+1}{2}\right)$$

**Proof:** This is obvious by noting that $R^*$ is the elements of $(\mathbb{Z}/p^n\mathbb{Z})$ that have multiplicative inverses, meaning the elements of $(\mathbb{Z}/p^n\mathbb{Z})$ that are not 0 mod $p$. In other words, everything that is in $P_p$ is included in $P_{p^n}$, for any $n$. Therefore,

$$N_k(p^n, R^*) = \text{number of non-zeros mod } p \text{ in first } (p^n)^k \text{ rows of } P_{p^n}$$

$$= \text{number of non-zeros mod } p \text{ in first } p^{nk} \text{ rows of } P_p$$

$$= N_{nk}(p, R^*).$$

Thus, applying Equation (3), we have

$$D(p^n, R^*) = \lim_{k \to \infty} \frac{\log(N_k(p^n, R^*))}{\log(p^{nk})} = \lim_{k \to \infty} \frac{\log(N_{nk}(p, R^*))}{\log(p^{nk})}$$

$$= \lim_{k \to \infty} \log\left(\frac{(\frac{p(p+1)}{2})^{nk} q_{R^*}(nk) + \sum_{\lambda_i \neq \frac{p(p+1)}{2}} |\lambda_i|^{nk} q_i(nk) [B_i \cos(\theta_i nk) + C_i \sin(\theta_i nk)]}{\log(p^{nk})}\right)$$

$$= \lim_{k \to \infty} \log\left(\frac{(\frac{p(p+1)}{2})^{nk} q_{R^*}(nk) + O\left(\left(\frac{p(p+1)}{2}\right)^{nk} q_{R^*}(nk)\right)}{\log(p^{nk})}\right),$$

35
and because $q_{R^*}(nk)$ is a positive constant (Lemma 4)

$$
\log \left( \frac{(p(p+1))^{nk} q_{R^*}(nk)}{1 + O\left( \frac{(p(p+1))^{nk} q_{R^*}(nk)}{2} \right)} \right)
= \lim_{k \to \infty} \frac{\log \left( \frac{(p(p+1))^{nk} q_{R^*}(nk)}{2} \right)}{\log(p^{nk})}

= \lim_{k \to \infty} \frac{\log \left( \frac{(p(p+1))^{nk} q_{R^*}(nk)}{2} \right)}{\log(p^{nk})} = \lim_{k \to \infty} \frac{n k \log \left( \frac{(p(p+1))^{nk}}{2} \right)}{n k \log(p)} + \frac{\log \left( q_{R^*}(nk) \right)}{n k \log(p)}

= \frac{\log \left( \frac{(p(p+1))}{2} \right)}{\log(p)} = 1 + \log \left( \frac{p + 1}{2} \right).
$$

\[36\]

3.3 Proving $D(m) = 1 + \log_{p_m} \left( \frac{p_m + 1}{2} \right)$

**Theorem 8.** If $m$ is a square-free integer, then

$$D(m) = 1 + \log_{p_m} \left( \frac{p_m + 1}{2} \right),$$

where $p_m$ is the largest prime divisor of $m$.

**Proof:** First, let us slightly alter our typical approach to calculating $D(m)$. Let

$$\nu_k(m) = \# \text{ of non-zero cells in the first } (p_m)^k \text{ rows of } P_m,$$

where $p_m$ is the largest prime divisor of $m$. Then we have

$$D(m) = \lim_{k \to \infty} \frac{\log(\nu_k(m))}{\log((p_m)^k)}. \quad (14)$$

36
First, let us examine $\nu_k(m)$. We will find that

$$
\nu_k(m) = N_k(p_m) + \sum_{p_i|m, p \neq p_m} O(N_{\alpha_i(k)}(p_i)),
$$

where $\alpha_i(k)$ is an integer sufficiently large enough so that

$$
p_i^{\alpha_i(k)-1} < p_m^k < p_i^{\alpha_i(k)}.
$$

From (1), we know that the $P_m$ triangle is the union of all the triangles of $m$’s highest $p^n$ divisors. If we let $m$ be square-free, then $P_m$ is the union of only $P_{p_i}$ triangles, where $p_i$ is prime and divides $m$. We also know by (1) that in order for an entry in $P_m$ to be non-zero, it needs to be non-zero in only one of $P_{p_i}$. So at a particular row, say $(p_m)^k$ (where $p_m$ is the largest $p_i$), the number of non-zeros in the first $(p_m)^k$ rows of $P_m$ is going to be the number of non-zeros in the first $(p_m)^k$ rows of each individual $P_{p_i}$, minus those that get overcounted (because it’s possible to be nonzero for multiple $p_i$.) More specifically, the number of non-zeros in the first $(p_m)^k$ rows of $P_m$ is the number of non-zeros in the first $(p_m)^k$ rows of $P_{p_m}$, plus the non-zeros in each other prime divisor’s triangle, minus the non-zeros that get over-counted by being non-zero for multiple primes. In other words:

$$
\nu_k(m) = \text{non-zeros from } P_{p_m} + \sum_{p_i|m, p \neq p_m} \text{non-zeros from } P_{p_i}
$$

$$
- \text{non-zeros from } P_{p_i} \text{ that are already counted in another prime}
$$

but

$$
\text{(non-zeros from } P_{p_i} \text{ that are already counted in another prime)}
$$
\[ \leq \text{(non-zeros from } P_{p_m}), \]

thus:

\[ \nu_k(m) = \text{non-zeros from } P_{p_m} + \sum_{p_i|m, p_i \neq p_m} O(\text{non-zeros from } P_{p_i}). \]

In the first \((p_m)^k\) rows, the contribution from \(P_{p_m}\) is \(N_k(p_m)\), while the contribution from any other \(p_i\)-divisor’s triangle is strictly smaller than some “further down” iteration of \(N_k(p_i)\) (see Figure 17 or refer back to Section 0.2.2).

![Figure 17: Example of how \(p_i\)’s contribution to \(\nu_k(m)\) is \(O(N_{\alpha_i(k)}(p_i))\).](image)

Thus:

\[ \nu_k(m) = N_k(p_m) + \sum_{p_i|m, i \neq m} O(N_{\alpha_i(k)}(p_i)). \quad (15) \]
Combining (14) and (15),

\[
D(m) = \lim_{k \to \infty} \frac{\log \left( N_k(p_m) + \sum_{p_i | m, i \neq m} O(N_{\alpha_i(k)}(p_i)) \right)}{\log(p_m^k)}
\]

\[= \lim_{k \to \infty} \frac{\log \left( N_k(p_m) \left( 1 + \frac{\sum_{p_i | m, i \neq m} O(N_{\alpha_i(k)}(p_i))}{N_k(p_m)} \right) \right)}{\log(p_m^k)}. \tag{16}
\]

We know \(N_k(p_m)\)'s leading term is \(\left( \frac{p_m(p_m+1)}{2} \right)^k\) and the leading term of \(N_{\alpha_i(k)}(p_i)\) is \(\left( \frac{p_i(p_i+1)}{2} \right)^{\alpha_i(k)}\). Thus, the top-right of the above expression will be several terms of the form

\[
O \left( \frac{\left( \frac{p_i(p_i+1)}{2} \right)^{\alpha_i(k)}}{\left( \frac{p_m(p_m+1)}{2} \right)^k} \right) = O \left( \frac{\left( \frac{p_i(p_i+1)}{2} \right)^{\alpha_i(k)}}{\left( \frac{p_m(p_m+1)}{2} \right)^k} \right)
\]

and because \(p_m\) is larger than all other \(p_i\), we have

\[
\lim_{k \to \infty} O \left( \frac{\left( \frac{p_i(p_i+1)}{2} \right)^{\alpha_i(k)}}{\left( \frac{p_m(p_m+1)}{2} \right)^k} \right) = 0.
\]

This is not obvious, but the minutia of the reasoning is lengthy, and given explicitly in Theorem 11 in the Appendix. Thus, referring back to (16), we have

\[
D(m) = \lim_{k \to \infty} \frac{\log(N_k(p_m))}{\log(p_m^k)} = D(p_m, R) = 1 + \log_{p_m} \left( \frac{p_m + 1}{2} \right).
\]

Therefore, so long as \(m\) is square-free,

\[
D(m) = 1 + \log_{p_m} \left( \frac{p_m + 1}{2} \right).
\]

\[
\blacksquare
\]
The only reason why we must limit this argument to square free numbers is that we still do not know the leading term of $N_k(p^n)$ for $n \in \mathbb{Z}^>^1$, and thus would not be able to factor out the dominant term like we did in (16). Further, we wouldn’t know how the ratio in the upper right of (16) would behave either. If it is true that $N_k(p^n)$ and $N_k(p)$ have the same leading term, then our proof for a general $m$ would be analogous to the one above.
4 Applications and Open Questions

4.1 Fractal Dimension for a Power of Prime Modulus

When this project was in its primordial stage, it seemed like it would go down one of two paths: the fractal dimension of particular residues in $P_p$, or the fractal dimension of all non-zero residues of $P_{p^n}$ for $n \in \mathbb{N}$. As the former was accomplished, the latter remains an open question.

We were able to attribute a fractal dimension to any non-empty subset of residues within $P_p$ by finding $N_k(p, \{r\})$ through repeated matrix multiplication. Can we use such an approach for modulus of the form $p^n$? While Granville does show how to find the growth triangle $T_{p^n}$ [19], the properties of $T_{p^n}$ differ from $T_p$, making statement (5)

$$N_k(p, \{r\}) = \sum_{l=1}^{p-1} (\text{# of } r\text{-cells in } l \cdot T_p)N_{k-1}(p, \{l\})$$

no longer hold, and therefore the matrix associated with $P_{p^n}$, $M_{p^n}$ not able to be constructed. Thus, the entirety of our approach to finding $D(p, R)$ cannot be extended to finding $D(p^n, \mathbb{Z}/p^n\mathbb{Z} - \{0\})$. However, we have reason to believe that $D(p^n, R) = 1 + \log_p\left(\frac{p+1}{2}\right)$, regardless of $n$. This is for two reasons.

The first line of reasoning stems from the fact that generated images of $P_{p^n}$ resemble their respective $P_{p^{n-1}}$ images, only more complex (copies of $P_p$ appear in previous empty space). We already know from section 3.2 that if $R^* = (\mathbb{Z}/p^n\mathbb{Z})^*$, then $D(p^n, R^*) = 1 + \log_p\left(\frac{p+1}{2}\right)$. For each $n$, if it is true that $P_{p^n}$ contains copies of $P_{p^{n-1}}$ in what would be $P_{p^{n-1}}$’s white spaces, then by reverse induction $P_{p^n}$ is the countably infinite union of disjoint copies of $P_n$. Would this mean the fractal dimension is the same as $D(p)$?
Though, the above approach is only for when our $R = \mathbb{Z}/p^n\mathbb{Z} - \{0\}$. If we want to show that $D(p^n, R) = 1 + \log_p \left( \frac{p^1}{2} \right)$, we would further need to show that non-zero cells in $P_{p^n}$ that are 0 in $P_p$ still disperse themselves in a similar manner as in $P_p$; in just the right way to keep the fractal dimension the same. This can be done noting that if it is true that $P_{p^n}$ contains copies of $P_{p^{n-1}}$ in what would be $P_{p^{n-1}}$’s white spaces, then that copy triangle is composed of residues that are 0 mod $p^{n-1}$. In other words, residues of the form

$$0 \cdot p^{n-1} \mod p^n, 1 \cdot p^{n-1} \mod p^n, 2 \cdot p^{n-1} \mod p^n, 3 \cdot p^{n-1} \mod p^n, \cdots, (p-1) \cdot p^{n-1} \mod p^n$$

which form an additive group of order $p$, with each element being bijectively mapped to an element of $\mathbb{Z}/p\mathbb{Z}$, making it have the same behavior as $P_p$. This doesn’t tell us every residue will appear, but it does tell us that if for each $k$, the $(p^k + 1)^{th}$ row of $P_{p^n}$ contains evenly separated non zero mod $p^k$ entries, then there must be copies of $P_{p^{k-1}}$ that originate from those points.

The second line of reasoning, though far from proof, results from observations during the rudimentary stages of this project. Quite early on, we used numerical simulations to manually count $N_k(p^n, \{r\})$ for the first 8 or so values of $k$. Because we noticed that $P_{p^n}$’s pattern iterates by powers of $p$ (rather than the expected powers of $p^k$), we let $N_k(p^n, \{r\})$ be the number of $r$-cells in the first $p^k$ rows. We used these to “guess and check” recurrences. Though proof of nothing, we noticed the following held for all data points we could collect for $m = 4$:

- $N_k(4, \{1\}) = 5N_{k-1}(4, \{1\}) - 6N_{k-2}(4, \{1\}) - 2(-1)^k$ giving an explicit relation of $N_k(4, \{1\}) = \frac{3^k}{2} + \frac{2 \cdot 3^k}{3} - \frac{1}{6}(-1)^k$.

- $N_k(4, \{2\}) = 3N_{k-1}(4, \{2\}) + 3^{k-2}$ giving an explicit relation of $N_k(4, \{2\}) =$
\[(k - 1)3^{k-2}.\]

- \(N_k(4, \{3\}) = 5N_{k-1}(4, \{3\}) - 6N_{k-2}(4, \{3\}) + 2(-1)^k\) giving an explicit relation of \(N_k(4, \{3\}) = \frac{3^k}{2} - \frac{2x2^k}{3} + \frac{1}{6}(-1)^k.\)

Assuming these are the actual explicit forms for \(N_k(4, \{r\})\), then notice that for any possible \(R\), \(N_k(4, R)\) will either have a leading term of the form

- \(k3^{k-2}\), in which case

\[
D(4, R) = \lim_{k \to \infty} \frac{\log(k3^{k-2} + O(k3^{k-2}))}{\log(2^k)} = \lim_{k \to \infty} \frac{\log(k3^{k-2}(1 + O(k3^{k-2})))}{\log(2^k)}
\]

\[
= \lim_{k \to \infty} \frac{\log(k3^{k-2})}{\log(2^k)} = \lim_{k \to \infty} \frac{\log(k)}{\log(2^k)} + \frac{\log(3^{k-2})}{\log(2^k)}
\]

\[
= \lim_{k \to \infty} \frac{\log(k)}{\log(2^k)} + \frac{\log(3^{k-2})}{\log(2^k)} = \lim_{k \to \infty} \frac{(k - 2) \log(3)}{k \log(2)} = \frac{\log(3)}{\log(2)}.
\]

- \(3^k\), in which case

\[
D(4, R) = \lim_{k \to \infty} \frac{\log(3^k + O(3^k))}{\log(2^k)} = \lim_{k \to \infty} \frac{\log(3^k(1 + O(3^k)))}{\log(2^k)}
\]

\[
= \lim_{k \to \infty} \frac{\log(3^k)}{\log(2^k)} = \lim_{k \to \infty} \frac{k \log(3)}{k \log(2)} = \frac{\log(3)}{\log(2)}.
\]

Assuming this is correct, this shows \(D(2^2, R) = D(2, R)\). Also, we noticed that \(N_k(4, \{1\}) + N_k(4, \{3\}) = N_k(2, \{1\})\). This is expected, because if a number is 1 mod 2, then it is either 1 mod 4 or 3 mod 4, and if a number is either 1 mod 4 or 3 mod 4, then it must be 1 mod 2. This trend continued in \(N_k(8, \{r\})\): \(N_k(8, \{1\}) + N_k(8, \{5\}) = N_k(4, \{1\})\), etc.
If it is true that $N_k(p^n, R)$ has a leading term of $\left(\frac{p(p+1)}{2}\right)^k$ as suggested by the previous computations and the results from Section 3.2, then not only is it true that $D(p^n, R) = 1 + \log_p\left(\frac{p+1}{2}\right)$, but also $D(m) = 1 + \log_{p_m}\left(\frac{p_m+1}{2}\right) \forall m \in \mathbb{Z}$. This is because if we apply an analogous process of calculating $D(m)$ to the one outlined in Section 3.3, when we get to the step similar to (16), we can still confirm the upper-right fraction will disappear to zero as $k$ goes to infinity. Thus, the rest of the proof remains the same, ending with $D(m) = 1 + \log_{p_m}\left(\frac{p_m+1}{2}\right)$, except with no limitations on $m$.

We also do not have the tools to determine $D(m, R)$ yet, even if $m$ is square-free. This is because while we do know that $\nu_k(m, R)$ will be determined by the largest prime divisor $p_m$ (because in (16) the upper-right $N_{a_i(k)}(p_i; R)$’s will still have dominant terms of the form $\left(\frac{p(p+1)}{2}\right)^k$), we don’t know how to count residues larger than $(p_m - 1)$ that appear in $\nu_k(m, R)$. For example, suppose we want to know $D(120, 62)$. The only 62’s that will matter will be the ones from the $P_{p_m}$-part of $P_m$. But how many are there? We know how to count the number of 2’s, as that is just $N_k(5, 2)$. Some subset of these 2’s mod 5 will be 62 mod 120, but we do not yet know how to count them.

### 4.2 Lacunarity of Residue-Patterns for a Prime Modulus

A non-intuitive result of the fact that $D(p, R) = 1 + \log_p\left(\frac{p+1}{2}\right)$ is that two fractals may be different, but have the same fractal dimension, and when superpositioned over each other, still have the same fractal dimension (e.g. $P_{3,1} \cap P_{3,2} = \emptyset$, $P_{3,1} \cup P_{3,2} = P_{3,\{1,2\}} = P_3$, but $D(3, 1) = D(3, 2) = D(3, \{1, 2\}) = D(3) = 1 + \log_3 2$). In fact, for a given prime $p$, there are $2^{p-1} - 1$ different fractals that can be extracted from $P_p$,
all of which have the same fractal dimension (for each of the $p-1$ possible $r$’s, either $r \in R$ or $r \notin R$, but $R \neq \emptyset$, making there be $2^{p-1} - 1$ possible sets $R$).

Another non-intuitive result lies in comparing the magnitude of $D$-values for certain fractals. Recall that the closer the fractal dimension is to a natural number, the closer that shape is to filling that dimensional space. With that in mind, it seems surprising that $P_{1995}$ and $P_{19,14}$ have the same fractal dimension ($1995 = 3 \cdot 5 \cdot 7 \cdot 19$, meaning its $D$-value is $1 + \log_{19}(10)$, the same as any $D(19, R)$, like $D(19, 14)$.) Referring to Figure 18, one can see that $P_{19,14}$ is a faint collection of dust in comparison to $P_{1995}$, but at further rows down, $P_{19,14}$ and $P_{1995}$ have the same dominating triangular pattern, although $P_{19,14}$ is still nearly invisible.

![Figure 18](image)

Figure 18: Illustration of how two wildly different fractals may have the same dimension. If $P_{19,14}$ is difficult to see, that is because there is barely anything there.

Perhaps even more perplexing is the fact that even though $P_{19,14}$ is practically nothing, it still has a larger $D$-value (and therefore fills more space) than $P_{210}$. $D(19, \{14\}) = 1 + \log_{19}(10) \approx 1.78$, and $D(210) = D(7) = 1 + \log_{7}(4) \approx 1.71$ (see Figure 19.)

All of this is evidence of the weakness of fractal dimension as a measure. While two very different fractals may have the same fractal dimension, they might have different lacunarity, a measure of how much space the fractal fills. Thus, a very worthwhile endeavor would be to investigate the lacunarity of these various $P_{m,R}$.
4.3 Fractal Dimension of Other Cellular Automata

There are other cellular automata that exhibit fractal properties, such as the so-called *Fibinomial Triangle* [21], and the triangle of Stirling Subset Numbers (*i.e.* Stirling Numbers of the Second Kind) [22]. With both, perhaps modified versions of Granville’s growth triangles [19] could be found, then the approach of this thesis of finding $M_p$ could be applied.
Conclusion

In conclusion, we were able to expand upon the fact that the image of non-zeros mod $p$ ($p$ prime) in Pascal’s Triangle will have a fractal dimension of $D(p) = 1 + \log_p \left( \frac{p+1}{2} \right)$. We were able to generalize this statement to various combinations of moduli and sets of residues. Keeping the modulus to be a prime, we found that no matter which non-empty collection of non-zero residues you choose to consider, the fractal dimension of the image is still $D(p, R) = D(p) = 1 + \log_p \left( \frac{p+1}{2} \right)$. If we allow the modulus to be of the form $p^n, n \in \mathbb{Z}^{>0}$, then we do not yet know the fractal dimension of all non-zero residues, but we do know that for $R^* = (\mathbb{Z}/p^n\mathbb{Z})^*$, then we have again $D(p^n, R^*) = D(p) = 1 + \log_p \left( \frac{p+1}{2} \right)$. Finally, for when the modulus is a square-free $m$, the image of non-zero residues mod $m$ in Pascal’s Triangle is $D(m) = 1 + \log_{p_m} \left( \frac{p_m+1}{2} \right)$, where $p_m$ is the largest prime divisor of $m$. 
Lemma 9. \( \frac{\log(x+1)}{\log(x)} \) is an increasing function for \( x > 1 \).

Proof: Let \( f(x) = \frac{\log(x+1)}{\log(x)} \). We know \( f(x) \) is increasing for \( x > 1 \) if

\[
f'(x) = \frac{\frac{\log(x)}{x+1} - \frac{\log(x+1)}{x}}{\log^2(x)} > 0
\]

for \( x > 1 \). We will show this. First, note the obvious inequality,

\[
\log(2) > \log \left(1 + \frac{1}{t}\right)
\]

for \( t > 1 \). Then we add \( 2 \log(2) \) to both sides

\[
\log(2) + 2 \log(2) > \log \left(1 + \frac{1}{t}\right) + 2 \log(2)
\]

\[
3 \log(2) > \log \left(4 \left(1 + \frac{1}{t}\right)\right)
\]

\[
\int_1^x 3 \log(2) \, dt > \int_1^x \log \left(4 \left(1 + \frac{1}{t}\right)\right) \, dt
\]

\[
(x - 1)3 \log(2) > x \log \left(\frac{4(x+1)}{x}\right) + \log \left(\frac{x+1}{16}\right)
\]

\[
3x \log(2) - 3 \log(2) > 2x \log(2) + x \log(x + 1) - x \log(x) + \log(x + 1) - 4 \log(2)
\]

\[
x \log(2) + \log(2) > x \log(x + 1) - x \log(x) + \log(x + 1)
\]

\[
x \log(x) > x \log \left(\frac{x+1}{2}\right) + \log \left(\frac{x+1}{2}\right)
\]

\[
x \log(x) > (x + 1) \log \left(\frac{x+1}{2}\right)
\]
\[
\frac{\log(x)}{x+1} > \frac{\log \left( \frac{x+1}{2} \right)}{x} \\
\log(x) \cdot \frac{x+1}{x} - \log \left( \frac{x+1}{2} \right) > 0 \\
\log(x) \cdot \frac{x+1}{x} = \frac{\log \left( \frac{x+1}{2} \right)}{\log^2(x)} > 0
\]

as needed. Thus, \( \frac{\log(x)}{\log(x)} \) is an increasing function for \( x > 1 \).

Lemma 10. For \( 2 \leq p_i < p_m \),

\[
\left( \frac{p_i(p_i + 1)}{2} \right)^{\log_{p_i(p_m)}(p_m)} < \frac{p_m(p_m + 1)}{2}
\]

Proof: Because of Lemma 9, and because \( p_i < p_m \),

\[
\frac{\log \left( \frac{p_i+1}{2} \right)}{\log(p_i)} < \frac{\log \left( \frac{p_m+1}{2} \right)}{\log(p_m)}
\]

\[
\frac{\log(p_m)}{\log(p_i)} < \frac{\log \left( \frac{p_m+1}{2} \right)}{\log \left( \frac{p_i+1}{2} \right)}
\]

\[
\log_{p_i(p_m)}(p_m) < \log \left( \frac{p_i+1}{2} \right) \left( \frac{p_m+1}{2} \right)
\]

Now use these as exponents

\[
\left( \frac{p_i+1}{2} \right)^{\log_{p_i(p_m)}(p_m)} < \left( \frac{p_i+1}{2} \right)^{\log \left( \frac{p_i+1}{2} \right) \left( \frac{p_m+1}{2} \right)} = \left( \frac{p_m+1}{2} \right)
\]
\[
 p_m \left( \frac{p_i + 1}{2} \right)^{\log_{p_i}(p_m)} < p_m \left( \frac{p_m + 1}{2} \right)
\]

\[
 p_i^{\log_{p_i}(p_m)} \left( \frac{p_i + 1}{2} \right)^{\log_{p_i}(p_m)} < p_m \left( \frac{p_m + 1}{2} \right)
\]

\[
 \left( \frac{p_i(p_i + 1)}{2} \right)^{\log_{p_i}(p_m)} < \frac{p_m(p_m + 1)}{2}
\]

as needed.

\[\blacksquare\]

**Theorem 11.**

\[
 \lim_{k \to \infty} \frac{N_{\alpha_i(k)}(p_i)}{N_k(p_m)} = \lim_{k \to \infty} \left( \frac{p_i(p_i+1)}{2} \right)^{\alpha_i(k)} \frac{\alpha_i(k)}{p_m(p_m+1)k} = 0.
\]

**Proof:** Note that \(\alpha_i(k)\) is defined as the integer such that

\[
 p_i^{\alpha_i(k)-1} < p_m^k < p_i^{\alpha_i(k)}.
\]

This implies

\[
 \alpha_i(k) - 1 < \log_{p_i}(p_m^k) < \alpha_i(k)
\]

\[
 \alpha_i(k) < k \log_{p_i}(p_m) + 1 < \alpha_i(k) + 1
\]

which then allows us to say

\[
 \lim_{k \to \infty} \frac{p_i(p_i+1)}{2}^{\alpha_i(k)} = \lim_{k \to \infty} O \left( \frac{p_i(p_i+1)^{k \log_{p_i}(p_m)+1}}{2} \right) \]

50
\[
\lim_{k \to \infty} \mathcal{O}\left[ \left( \left( \frac{p_i(p_i+1)}{2} \right)^{\log_{\alpha_i}(p_m)} \right)^k \left( \frac{p_i(p_i+1)}{2} \right)^k \right]
\]

\[
= \lim_{k \to \infty} \mathcal{O}\left[ \left( \frac{p_i(p_i+1)}{2} \right)^{\log_{\alpha_i}(p_m)} \left( \frac{p_i(p_i+1)}{2} \right)^k \right]
\]

\[
= \lim_{k \to \infty} \left[ \left( \frac{p_i(p_i+1)}{2} \right)^{\log_{\alpha_i}(p_m)} \left( \frac{p_i(p_i+1)}{2} \right)^k \right] = 0
\]

by Lemma 10.
References


Author’s Biography

Elliot Ossanna is a native “downeast” Mainer. He loves driving: when he lived in Franklin, he went to school in Blue Hill, but when living in MDI, went to school in Sullivan. His hobbies include guitar, tennis, knitting, soccer, skateboarding, hiking, juggling, and Magic the Gathering. Perhaps his biggest weakness is that he finds *everything* interesting; it is exhausting.

His primary area of interest, outside of mathematics, is education. Having received a wonderful education at both an alternative Waldorf elementary school, but also a notoriously “worst-performing” high school, Elliot has lived both extremes of the public and private education systems. He often considers strengths and the shortcomings of each, and if and how a “perfect” system could be attained.