"When two objects, qualities, classes, or attributes, viewed together by the mind, are seen under some connection, that connection is called a relation."

... Augustus De Morgan

**Section 3.1  Relations**

**Purpose of Section**  
To introduce the concept of a relation between objects, the objects being almost anything one can imagine. This section acts as the backdrop for the three important relations studied in the following sections: the order relation, the equivalence relation, function relation, and graphs.

**Introduction**

In mathematics the word relation is used to show the existence or nonexistence of certain connections pairs of objects, taken in definite order, as in

- "is less than"
- "is perpendicular to"
- "is greater than"
- "is parallel to"
- "is a subset of"
- "divides"
- "is congruent to"
- "is equivalent to"

There are several types of relations, such as order relations, equivalence relations, functions, and graph/networks, all of which play important roles in various areas of mathematics, and are studied in this chapter.

**Cartesian Product of Sets**

Before formally defining a relation, we must introduce the concept of the Cartesian product of two sets.

**Definition  Cartesian Product:**
Let $A$ and $B$ be arbitrary sets. The **Cartesian product** of $A$ and $B$, denoted $A \times B$ and read " $A$ cross $B$ " is the set of ordered pairs

$$A \times B = \{(a,b) : a \in A, b \in B \}.$$ 

That is $A \times B$ is the set of all ordered pairs $(a,b)$, where the first coordinate $a$ is taken from $A$ and the second coordinate $b$ is taken from $B$.

**Example 1**  
Let $A = \{1, 2\}$ and $B = \{1, 3, 4\}$, then
\[ A \times B = \{(1,1),(1,3),(1,4),(2,1),(2,3),(2,4)\} \]

If we represent the points \((a,b)\) as points in the \(xy\)-Cartesian plane, the Cartesian product \(A \times B\) consists of the points displayed in Figure 1.

![Cartesian Product \(A \times B\) Figure 1](image)

Order is important for Cartesian products. For the same sets \(A, B\), we have

\[ B \times A = \{(1,1),(1,2),(3,1),(3,2),(4,1),(4,2)\} \]

which illustrates that in general \(A \times B \neq B \times A\). The points in the Cartesian product \(B \times A\) are reflections of the points in \(A \times B\) through the line \(y = x\).

![Cartesian Product \(B \times A\) Figure 2](image)

**Margin Note:** Do not confuse the ordered pair \((0,1)\) from the open interval on the real line \((0,1) = \{x: 0 < x < 1\}\).
Cartesian products on more than two sets are defined accordingly. For instance, for three sets \( A, B, \) and \( C \) we would define

\[ A \times B \times C \equiv \{(a, b, c) : a \in A, b \in B, c \in C\}. \]

When the coordinates are real numbers, the ordered triple \((a, b, c)\) can be interpreted as points in three-dimensional space.

The Cartesian product interacts with the union and intersection of sets by the following properties.

**Example 2** If \( A, B \) and \( C \) are sets, then

\[
\begin{align*}
\text{a) } A \times (B \cup C) &= (A \times B) \cup (A \times C) \\
\text{b) } A \times (B \cap C) &= (A \times B) \cap (A \times C) \\
\text{c) } (A \times B) \cap (C \times D) &= (A \cap C) \times (B \cap D) \\
\text{d) } (A \times B) \cup (C \times D) &= (A \cup C) \times (B \cup D)
\end{align*}
\]

**Proof:**

We prove a). The proofs of b), c) and d) are left to the reader. To prove part a) note that both sets \( A \times (B \cup C) \) and \((A \times B) \cup (A \times C)\) are sets of ordered pairs, where their elements are of the form \((x, y)\). Normally, a proof showing the equality of two sets requires two separate proofs, where one proves each set is a subset of the other, but in this case we can go “forward” and “backwards” at the same time. Watch carefully.

\[
\begin{align*}
(x, y) \in A \times (B \cup C) &\iff x \in A \text{ and } y \in B \cup C \\
&\iff x \in A \text{ and } (y \in B \text{ or } y \in C) \\
&\iff (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \\
&\iff (x, y) \in A \times B \text{ or } (x, y) \in A \times C \\
&\iff (x, y) \in (A \times B) \cup (A \times C).
\end{align*}
\]
Since all the logical connectives in the proof are “if and only if” you can start either at the top or bottom of the argument and work to the other end. Many proofs showing the equality of sets work in this manner. Figure 2 provides a visualization of this identity although in general $A, B, C$ need not be subsets of the real numbers.

**Margin Note:** Readers are already familiar with the most popular Cartesian product, which is $\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R}$, the Cartesian plane.

In Figure 3 we have drawn some typical Cartesian products.
A (binary) relation is a function that assigns truth values (true or false) to two individuals, which can be numbers, sets, functions and so on. When the value assigned is true, we say the individuals are related, like if the relation were “is less than” then the pair of numbers \((2,3)\) would be assigned “true” since \(2 < 3\). The reader is already familiar with many relations, such as the equal relation “=” which would assign a truth value to the pair \((3,3)\) but not the pair \((3,4)\). Relations also occur outside of mathematics as the following relations.

- John is taller than Jerry
- Susan is the daughter of John
- Harry loves Susan
- Des Moines is a city in Iowa
Relations can be between numbers and themselves, like \(2 < 3\), between people and themselves, and between a city and a state.

- John \(R\) Jerry \(\quad R = \text{"is taller than"}\)
- \(2R3\) \(\quad R = \text{"<"}\)
- Susan \(R\) John \(\quad R = \text{"is the daughter of"}\)

This motivates the general definition of a relation.

**Definition: Relation**

Let \(A\) and \(B\) be sets and \(A \times B\) the Cartesian product. A **binary relation** from the set \(A\) to the set \(B\), which we denote by \(R\), is a subset of \(A \times B\). If \((x, y) \in R\) we denote this by writing \(xRy\), where we say “\(x\) and \(y\) are related (relative to \(R\)). If \((x, y) \notin R\) we say that \(x\) and \(y\) are not related, this denote this by \(\not{xRy}\). In case \(A = B\) a relation from \(A\) to \(A\) is called a relation on \(A\). The elements in a relation \(R\) is called the **graph** of the relation.

You can think of a binary relation as an operation that assigns either \(T\) or \(F\) (or 1 or 0) to the pair of objects. If the value is true we say the objects are related, if the value is false the objects are not related.

**Margin Note:** The definition of a relation can be a bit confusing at first, thinking of a relation as both a set (i.e. subset of a Cartesian product) and as a relation between things, like “\(=\)”, “\(<\)”, “loves” and so on. Just realize that if a point \((x, y) \in R\) then we often denote this by writing \(xRy\).

The following examples will familiarize you with relations.

**Example 3 (Typical Relation)**

Consider a set \(A\) of five students and a set \(B\) of four movies:

\[
A = \{\text{Mary, John, Ann, Sally, Jim}\}
\]

\[
B = \{\text{Crash, Batman, Star Wars III, Sin City} \}
\]

and consider the relation \(R \subseteq A \times B\) defined by
and consists of the 10 \( x \)'s drawn in the 20 boxes in Table 1. The elements of \( R \) describe those students who like a certain movie. Since (John, Sin City) ∈ \( R \), this means John likes the movie Sin City. Since (Ann, Star Wars IX) ∉ \( R \), this means Ann does not like Star Wars IX. The subset \( R \subseteq A \times B \) is a relation on \( A \times B \) (read “relation from \( A \) to \( B \)”).

<table>
<thead>
<tr>
<th>Sin City</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Star Wars IX</td>
<td>( x )</td>
<td>( x )</td>
</tr>
<tr>
<td>Batman</td>
<td>( x )</td>
<td>( x )</td>
</tr>
<tr>
<td>Crash</td>
<td>( x )</td>
<td>( x )</td>
</tr>
<tr>
<td>Mary</td>
<td>John</td>
<td>Ann</td>
</tr>
</tbody>
</table>

Relation “likes movie” on \( \{\text{Students}\} \times \{\text{Movies}\} \)

Table 1

**Example 4 Identify the Relation**

The Cartesian product of \( A = \{1, 2\}, B = \{1, 3, 4\} \) consists of the \( 2 \cdot 3 = 6 \) pairs of points

\[
A \times B = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}
\]

A relation on \( A \times B \) is any one of the \( 2^6 = 64 \) subsets of \( A \times B \). A popular relation (do you recognize it?) from \( A \) to \( B \) would be

\[
R = \{(1,2),(1,3),(2,3)\} \subseteq A \times B
\]

In other words, “1 is related to 2”, 1 is related to 3” and “2 is related to 3”.

Do you recognize the relation now? We would also denote this relation by writing

\[
1R2 \text{ since } (1,2) \in R\quad \text{more commonly } (1 < 2)
\]

\[
1R3 \text{ since } (1,3) \in R\quad \text{more commonly } (1 < 3)
\]

\[
2R3 \text{ since } (2,3) \in R\quad \text{more commonly } (2 < 3)
\]

In this example the relation \( R \) is the relation “is strictly less than” which we normally denote by ”\(<\)”. We also write \( 1R1 \) or \( \cancel{1R1} \) (1 is not related to itself) since 1 is not strictly less than 1. The relation is illustrated by a “directed graph” drawn in Figure 4.
Example 5  Subset and Membership Relations

The subset "\( \subseteq \)" is a relation on the power set \( P(A) \) of a set \( A \), and "\( \in \)" is a relation from the set \( A \) to its power set \( P(A) \).

Example 6  A More Familiar Relation

Often \( A \) and \( B \) are subsets of the real numbers \( \mathbb{R} \) such as

\[
A = B = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} = [0,1]
\]

Here the Cartesian product is the unit square

\[
[0,1] \times [0,1] = \{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq 1 \}
\]

which is shaded in Figure 5. A common relation from \( [0,1] \) to itself is

\[
R = \{(x,y) \in [0,1] \times [0,1] : y = x^2 \}
\]

which the reader will recognize as the graph of the parabola \( y = x^2 \) drawn in Figure 5. We would call this relation the “squaring” relation and since \( (1/2,1/4) \in R \) we would say 1/2 is related to 1/4, the denote this relationship
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by \((1/2)^2 = 1/4\). Later, we will denote function relations by \(f, g,...\) and so this relation would be denoted by

\[
(f, g) = \{(x, y) \in [0,1] \times [0,1] : y = x^2\}.
\]

A Function Relation

Figure 5

Margin Note: A relation \(R\) from the real numbers \(\mathbb{R}\) to itself is a subset of the Cartesian plane \(\mathbb{R} \times \mathbb{R}\). For any point \((x, y) \in R\) we say that the first coordinate \(x\) is related to the second coordinate \(y\).

Inconsistency in Notation: Although relations are sets, we often do not “think” of them that way. For instance, we express the inequality relation \(\leq\) by writing \(xRy\) as in \(2 \leq 3\). The equivalent set notation for \(\leq\) would be the awkward looking \((2,3) \in \leq\). (You would probably scratch your head if inequalities were written like that!) Also, we seldom use the \(xRy\) “form” to express function relations. They have their own notion. For example, we normally write functions like \(y = f(x) = x^2\). This relation written as \(xRy\) would be \(xRf(x)\), again an awkward looking expression for most of us. Sometimes, however, it is convenient to express functions in set notation, as when we write \((x, x^2) \in f\). This notation has merit since it interprets a function in terms of what we normally call the graph of the function.

Higher Order Relations: In set theory and logic a \(n\)-ary is a relation that assigns a truth value (true or false) to \(k\) members of a given set. A ternary relation is a relation \(R(a, b, c)\) that assigns a truth value to three members. For example the statement that for nonzero integers \(a, b, c\)
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\[a^3 + b^3 = c^3\]

is a ternary relation on \(\mathbb{Z}\), which according to Andrew Wiles\(^1\) is false.

Domain and Range of a Relation

Two important aspects of relations are the domain and range of a relation.

**Definition: Domain and Range** Let \(R\) be a relation from \(A\) to \(B\). The **domain** of the relation is the set

\[\text{Dom}(R) = \{x \in A : \exists y \in B \text{ such that } xRy\}.\]

The **range** of the relation \(R\) is

\[\text{Range}(R) = \{y \in B : \exists x \in A \text{ such that } xRy\}.\]

*In Plain English:* The domain of a relation \(R\) from \(A\) to \(B\) is the set of first coordinates of the ordered pairs in \(R\). The range of a relation \(R\) is the set of second coordinates of the ordered pairs of \(R\). By definition, we have \(\text{Dom}(R) \subseteq A\), \(\text{Range}(R) \subseteq B\).

**Example 7** Let \(A = B = \mathbb{R}\) (real numbers). We define a relation on \(\mathbb{R} \times \mathbb{R}\) by \(R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 \leq 1\}\) which are points on and inside the unit circle in the Cartesian plane. Here

\[\text{Dom}(R) = \{x \in \mathbb{R} : -1 \leq x \leq 1\}\]

\[\text{Range}(R) = \{y \in \mathbb{R} : -1 \leq y \leq 1\}\].

**Margin Note:** If \((a, b) \in R\), then

i) \(a \in \text{Dom}(R)\)

ii) \(b \in \text{Range}(R)\)

iii) \(a\) is related to \(b\)

See Figure 6.

\(^1\) English mathematician Andrew Wildes verified Format’s Last Theorem in the affirmative by proving that among nonzero integers \(a, b, c\) the equation \(a^n + b^n = c^n\) has no solutions for integers \(n \geq 3\).
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Inverses and Compositions

Two common ways to construct new relations from old ones are inverse relations and composition of relations.

**Definition**

If \( R \) is a relation from \( A \) to \( B \), then the inverse of \( R \) is defined as

\[
R^{-1} = \{(y, x) : (x, y) \in R\}.
\]

For example the inverse of \( R = \{(1, 2), (3, 5), (4, 1)\}\) is \( R^{-1} = \{(2, 1), (5, 3), (1, 4)\}\).

**Example 8**

Let \( A = \{1, 2, 3, 4\} \) with a relation on \( A \) given by

\[
R = \{(1, 2), (1, 3), (2, 4), (3, 4)\}.
\]

The inverse of \( R \) is \( R^{-1} = \{(2, 1), (3, 1), (4, 2), (4, 3)\}\). Both \( R \) and \( R^{-1} \) are drawn in Figure 7. Note that the graph of \( R^{-1} \) is the mirror image or reflection of \( R \) across the line \( y = x \).
Example 9 (Inverse of a Curve)
Given the relation on \( \mathbb{R} \) defined by \( R = \{(x, y) : y = x^2\} \) the inverse is \( R^{-1} = \{(x, y) : x = y^2\} \) which we have drawn in Figure 8.

Composition of Relations
The composition of two (or more) relations is similar to the composition of functions of which the reader is familiar. Recall that the composition of functions \( f(x) \) and \( g(x) \), written \( f \circ g \), is defined by \( (f \circ g)(x) = f(g(x)) \) for all \( x \) in the domain of \( g \) and \( g(x) \) in the domain of \( f \). The generalization of this definition to relations is as follows.
**Definition:** Let $R$ be a relation from $A$ to $B$ (i.e. a subset of $A \times B$) and $S$ a relation from $B$ to $C$ (i.e. a subset of $B \times C$). The **composite (or composition)** of $R$ and $S$ is the relation

$$S \circ R = \{(a,c) : \text{there exists } b \in B \text{ such that } (a,b) \in R \text{ and } (b,c) \in S\}.$$ 

**Example 10 (Composition of Relations)**

Let $A = \{1,2,3,4\}$, $B = \{a,b,c\}$, $C = \{\text{cat, dog, horse}\}$ and define the relations

$$R = \{(1,a),(1,c),(2,a),(3,b),(4,b)\}$$

$$S = \{(a,\text{dog}),(b,\text{horse}),(b,\text{cat}),(c,\text{dog}),(c,\text{horse})\}$$

which are illustrated in Figure 9. The composition $S \circ R$ in this diagram consists of all “paths” starting from $A$ and ending at $C$, which are

$$S \circ R = \{(1,\text{dog}),(2,\text{dog}),(3,\text{cat}),(3,\text{horse}),(4,\text{cat}),(4,\text{horse})\}$$
Problems

1. **(Four Basic Cartesian Products)** Let \( A = \{1, 2, 3\}, B = \{a, b\} \). Find the following Cartesian products.

   a) \( A \times B \)
   b) \( B \times A \)
   c) \( A \times A \)
   d) \( B \times B \)

2. **(Cartesian Products)** For each pair of sets \( A \) and \( B \) find the Cartesian products \( A \times B \) and \( B \times A \).

   a) \( A = \{0, 2\}, B = \{-1, 0, 1\} \)
   b) \( A = \{a, b\}, B = \{a, b, c\} \)
   c) \( A = \mathbb{R}, B = \mathbb{N} \)
   d) \( A = \mathbb{Z}, B = \mathbb{N} \)
   e) \( A = \mathbb{R}, B = \{-1, 0, 1\} \)
   f) \( A = \mathbb{C}, B = \mathbb{C} \)

3. **(Graphing a Relation)** Sketch the following relations.

   a) \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\} \)
   b) \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \sin x\} \)
   c) \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y^2\} \)
   d) \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq 1, |y| \geq 1\} \)
   e) \( R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \equiv 0 (\text{mod } 3), y \equiv 1 (\text{mod } 3)\} \)
   f) \( R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \text{ divides } y\} \)

4. **(Algebra of Relations)** Suppose \( A = [0, 3], B = [2, 5], C = [1, 4] \) are closed intervals on the real \( \mathbb{R} \). Sketch the following relations on \( \mathbb{R} \times \mathbb{R} \).

   a) \( (A \cup B) \times C \)
   b) \( (A \cap B) \times C \)
   c) \( (A \times B) \cup (A \times C) \)
   d) \( A \times (A \cap C) \)
   e) \( A \times (A \cap B) \)
5. (Naming a Relation) Find the inverse relations of the following relations on \( A = \{1, 2, 3\} \). Plot the points of the relation and give each relation a name which best describes it.

a) \( R = \{(1,1),(2,2),(3,3)\} \)

b) \( R = \{(1,2),(1,3),(2,3)\} \)

c) \( R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\} \)

d) \( R = \{(2,1),(3,1),(3,2)\} \)

6. (Inverse Relations on \( \mathbb{R} \)) Find and graph the inverse relations of the following relations on \( \mathbb{R} \).

a) \( R = \{(x,y): y = \log_a x\} \)

b) \( R = \{(x,y): 2x + y = 0\} \)

c) \( R = \{(x,y): xy = 1\} \)

d) \( R = \{(x,y): |x| + |y| \leq 1\} \)

e) \( R = \{(x,y): |x - y| \leq 1\} \)

f) \( R = \{(x,y): |x + y| \leq 1\} \)

7. (Cartesian Product Identities) Prove the following.

\[
\begin{align*}
a) A \times (B \cap C) &= (A \times B) \cap (A \times C) \\
b) (A \times B) \cap (C \times D) &= (A \cap C) \times (B \cap D) \\
c) (A \times B) \cup (C \times D) &= (A \cup C) \times (B \cup D)
\end{align*}
\]

8. (Number of Relations) If a set \( A \) has \( m \) elements and \( B \) has \( n \) elements then show there are \( 2^{mn} \) distinct relations from \( A \) to \( B \).

9. (Identity Relation) For any set \( A \), the set \( I_A = \{(x,x): x \in A\} \) is called the identity relation on \( A \). For the set \( A = \{1,2,3,4\} \), what is the domain and range of \( I_A \) and draw its graph.

10. (Properties of Relations) Determine whether the following relations are reflexive, anti symmetric, or transitive.

a) \((x, y) \in R \iff \text{the greatest common divisor of } x \text{ and } y \text{ is } 1\)
b) \((x, y) \in R \iff x < y\)

c) \((x, y) \in R \iff 5 \text{ divides } x - y\)

d) \((x, y) \in R \iff x \geq y\)

11. Give examples of relations on the \(A = \{1, 2, 4, 5\}\) having the following properties.

   a) Reflexive, antisymmetric, not transitive
   b) Reflexive, symmetric, not transitive
   c) Reflexive, antisymmetric, transitive
   d) Not reflexive, symmetric, transitive