Section 2.5 Uncountable Sets

Purpose of Section To introduce the concept of uncountable sets. We present Cantor's proof that the real numbers are uncountable. We also show that the cardinality of the plane is the same as the cardinality of the real numbers.

Introduction

It easy to see there can't be *more* natural numbers than real numbers since the identity function $f(n) = n$ is a one-to one correspondence from $\mathbb{N}$ into a *subset* of $\mathbb{R}$. The question is, is there a one-to-one correspondence from the natural numbers *onto* the real numbers? If so, that says there are the *same* number of natural numbers as real numbers. If not, it says there are *fewer* natural numbers than real numbers; i.e. $|\mathbb{N}| < |\mathbb{R}|$. But it is one thing to find a bijection from one set to another, it is quite another to prove that one does *not* exist. But this is exactly what Candor did. Using a proof by contradiction, Cantor showed there was no one-to-one correspondence from $\mathbb{N}$ to $\mathbb{R}$, and hence the natural numbers have a *smaller* cardinality than the real numbers. A set that has a strictly larger cardinality than the natural numbers is called **uncountable**.

Cantor's proof that the real numbers has a larger cardinality than the natural numbers is called the **Cantor Diagonal argument**. The proof and its results so amazed himself that he wrote to his good friend Richard Dedekind saying, "I see it but I don't believe it."

Theorem 1 The real numbers $\mathbb{R}$ are uncountable.

Proof: The proof is by contradiction. In other words, we assume that $\mathbb{R}$ is countable, which means we can match the natural numbers $1,2,3,\cdots$ with the real numbers. We will show that no matter *how* we list the real numbers, any such listing will miss some natural numbers, which contradicts the assumption the real numbers are countable. To do this consider a sample list of real numbers shown in Figure 5 expressed in decimal form[1] which we try to match with the natural numbers $1,2,3,\cdots$.

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[1] Every real number can be expressed uniquely in decimal form $a_0.a_1a_2a_3...$ where a_0 is an integer and the numbers $a_1, a_2...$ after the decimal are integers between $0 \le a_i \le 9$, provided the convention is made that if the decimal expansion ends with an infinite string of 9's , such as 0.499999… which is the same as 0.5, the expansion is modified by raising by 1 the last digit before the 9's and changing all the 9's to 0's. Making this convention provides a one-to-one correspondence between the real numbers and their decimal expansion.

1	⟷	2.197204817...
2	⟷	14.536613809...
3	⟷	0.497310123...
4	⟷	292.275818831...
5	⟷	12.002200025...
6	⟷	1.999902681...
⋮		⋮

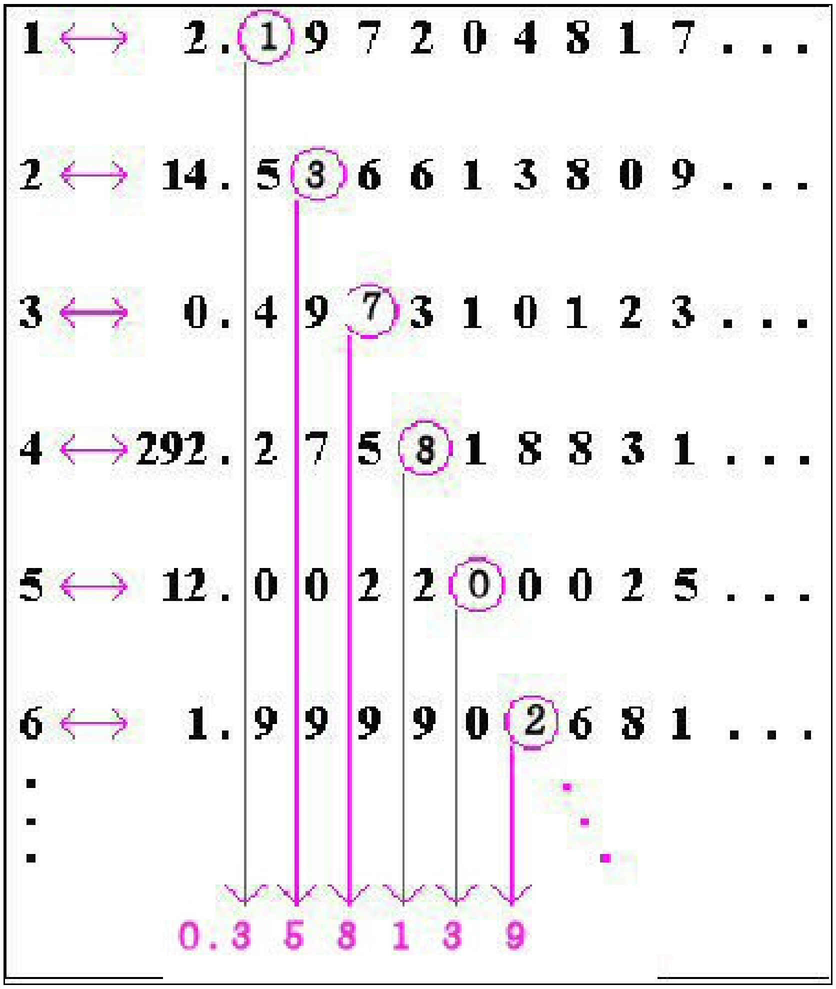
Cantor's hypothesized one-to-one correspondence
between the whole numbers and the real numbers.
Figure 5

If we can associate natural numbers with the real numbers, this means every natural number will be in the left column and ever real number in the right column. Cantor shows any such listing leads to a contradiction by creating a *rogue* real number that is not in the table, no matter what listing of real numbers. Hence the assumption that the real numbers is countable leads to a contradiction, and hence the reals are uncountable.

The strategy to find Cantor's rogue real number is to find a number $0.a_1a_2a_3...$ (which we write in decimal form) that can not be associated with any natural number. Keeping an eye on the list of numbers in Figure 6, Cantor picks the picks the first digit a_1 after the decimal point *different* from the first digit after the decimal in $2.\underline{\mathbf{1}}97204817\cdots$ (i.e not[2] 1). Doing this Cantor's rogue number could begin with $0.3\cdots$ (anything but 1). Next, Cantor chooses the second digit in the rogue number anything different from the second digit after the decimal of $14.5\underline{\mathbf{3}}6613809\cdots$ (i.e. not 3). We arbitrarily pick 5 for the second digit, giving us 0.35. Continuing this process, working down the *diagonal* of the table, Cantor might pick the first six digits of the rogue number to be $0.358139\cdots$. Continuing this process indefinitely, Cantor obtains a real number *not* in the table since the rogue number differs in *at least one* digit

[2] We have put this number in bold type and underlined it for your convenience

from each number in the list. But this contradicts the fact that the natural and real numbers can be placed in a one-to-one correspondence. Hence, they *cannot* be placed in one-to-one correspondence and since the natural numbers are a subset of the reals, and since the natural numbers are a subset of the real numbers, the real numbers must have a larger cardinality or "size" than the whole numbers. ▮



Cantor's diagonalization process.
Figure 6

Cantor now had two infinities, the countable infinity of the natural numbers, called $\aleph_0$, and a new infinity of the real numbers. Since he didn't know if the cardinality of the real numbers was the "next" larger infinity after $\aleph_0$, he denoted the cardinality of the reals by c, meaning the **cardinality of the continuum**. Hence, we say that the real numbers, and any set equivalent to the real numbers, has cardinality c or is **uncountable**.

Note: Roughly speaking an uncountable set has so many points it cannot be put in a sequence.

Example 1 (Another Uncountable Set)

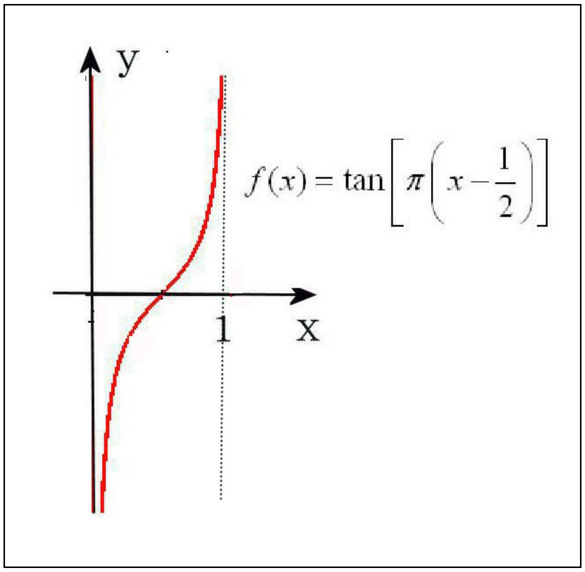
Show there are the same number of real numbers as there are numbers in the interval $(0,1)=\{x\in\mathbb{R}: 0<x<1\}$.

Solution

Define the function $f:(0,1)\to\mathbb{R}$ by

$$f(x)=\tan\left[\pi\left(x-\frac{1}{2}\right)\right],\ 0<x<1$$

which is a one-to-one correspondence between (0,1) and $\mathbb{R}$. Hence $(0,1)\approx\mathbb{R}$ and so the open interval (0,1) also has cardinality c. Figure 7 gives a graphical representation of this correspondence.



Correspondence Between (0,1) to $\mathbb{R}$

Figure 7

Sometimes we can graphically display a one-to-one correspondence between sets as in the following example.

Margin Note: The "lazy eight" symbol "∞" does not represent the number infinity; it is simply a symbol used to denote that a set of real numbers is unbounded, such as $(a,\infty),(-\infty,b),(-\infty,\infty)$ and so on.

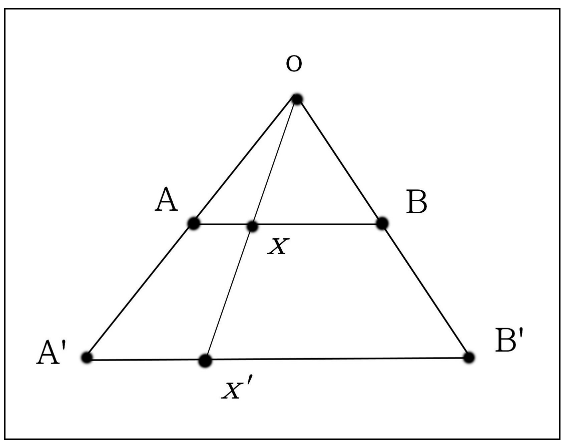
Example 3 (More and More Uncountable Sets)

Show that every interval (open or closed) of real numbers has the cardinality of the continuum c.

Solution

We know from Example 1 that the interval (0,1) has cardinality c, and Figure 8 gives a visual one-to-one correspondence between line segments of different length. Any two line segments $A'B'$ and AB of different lengths can be placed in one-to-one correspondence by the bijection $x \leftrightarrow x'$ illustrated in the drawing. Thus, all open (or closed) intervals on the real line are equivalent[3]. Even the very large interval $\left(-10^{100}, 10^{100}\right)$, which is longer than one can comprehend, and the tiny interval $\left(-10^{-100}, 10^{-100}\right)$ whose length is smaller than the width of an electron, both contain the same number of elements.

Margin Note: All intervals of the form (a,b), $[a,b]$, $[a,b)$, $(a,b]$, $(-\infty,b]$, $[a,\infty)$ have cardinality c.



The shorter segment AB contains as many points as the longer $A'B'$
Figure 8

[3] One can also show that all intervals

> **Margin Note:** The lazy figure eight symbol "∞" does not represent the number infinity; it is simply a symbol used to denote that a set of real numbers is unbounded, such as $(a,\infty),(-\infty,b),(-\infty,\infty)$ and so on.

After Cantor's discovered there are more real numbers than natural numbers, he set about to find even *larger* sets, and so he turned his attention to points in the plane, where he felt there would be larger sets. Cantor worked from 1871 to 1874 to prove this theorem false.

Theorem 2 (Cardinality of the Plane) The set of points in a square has the same cardinality as the points on one of its edges.

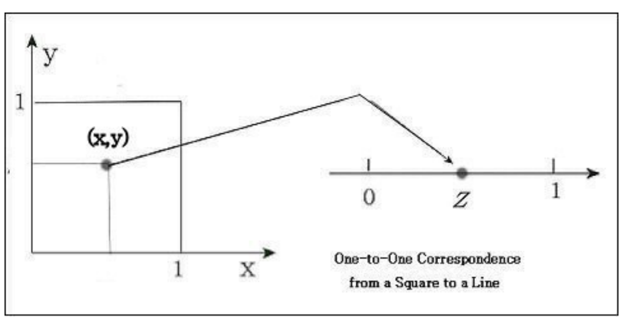
Proof: Without loss of generality, we take the square to be the unit square

$$S=(0,1)\times(0,1)\equiv\{(x,y):0<x<1,\ 0<y<1\}$$

in the Cartesian plane and show that every point $(x,y)\in S$ can be mapped in a one-to-one fashion onto a point z between 0 and 1. That is

$$(x,y)\leftrightarrow z$$

See Figure 9.



Correspondence from the Unit Square to the Unit Interval
Figure 9

To see how this mapping is constructed, suppose the point is $(x, y) = (0.2573\ldots, 0.3395\ldots)$. We then alternately interlace the digits of x and y, getting the single real number $z = 0.23537935\ldots$ which is a real number between 0 and 1. Conversely, for any real number, say $z = 0.34648391\ldots$ we can construct a unit point in the square, namely $(x, y) = (0.3689\ldots, 0.4431\ldots)$. The manner in which this mapping is defined, it maps different points (x, y) in the plane to different real numbers z. That is

$$(x_1, y_1) = (x_2, y_2) \Rightarrow z_1 = z_2$$

It is also clear by the way the mapping is defined that every real number z between 0 and 1 has a unique preimage point (x, y) in the square. Hence, we have found a one-to-one correspondence between the points inside the unit square S and the open interval $(0,1)$ ∎

Summary:

Countable Infinite sets (cardinality $\aleph_0$): $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}\times\mathbb{N}$
Uncountable sets (cardinality c) $\mathbb{R}, [a,b], (a,b), \mathbb{R}^2, \mathbb{R}-\mathbb{Q}$

There are other sets the reader has seen in undergraduate mathematics. The n-dimensional Euclidean space $\mathbb{R}^n$ has cardinality c, the set of all sequences of real numbers has cardinality c, the set of continuous function defined on an interval has a *larger* cardinality.

Cardinality of the Irrational Numbers: The interval $(0,1)$ is the disjoint union of the rational and irrational numbers inside $(0,1)$. We have seen that the interval $(0,1)$ has cardinality c and that the rational numbers has cardinality $\aleph_0$. That means the irrational numbers in the interval $(0,1)$ has cardinality c, the same as the entire interval. In other words, there are more irrational numbers than rational numbers.

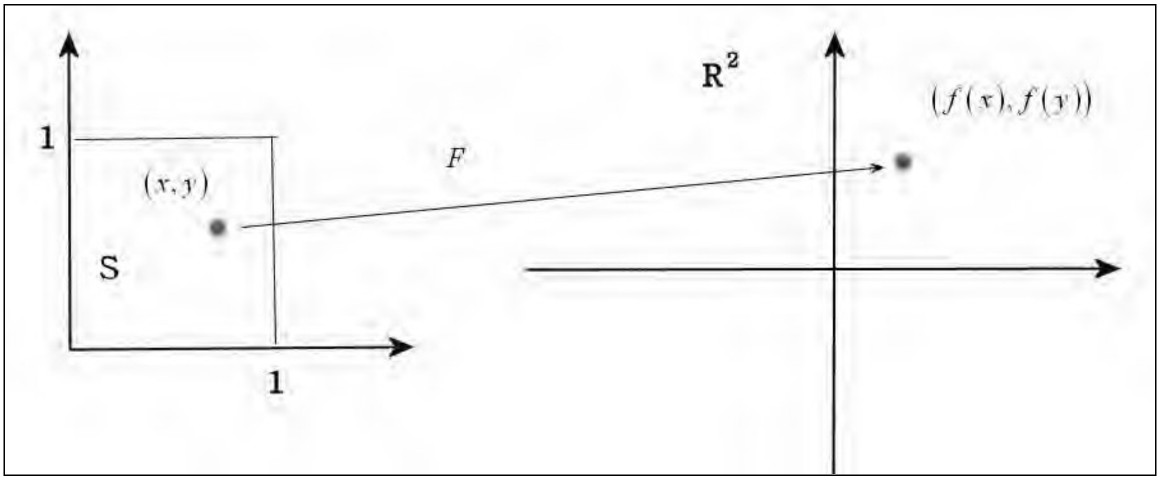
Example 4 Find a one-to-one correspondence between the open square

$$S = \{(x, y) : 0 < x < 1, 0 < y < 1\}$$

and the plane

$$\mathbb{R}^2 = \{(x, y) : -\infty < x < \infty, -\infty < y < \infty\}.$$

Solution



Consider the function

$$f(x) = \tan\left[\pi\left(x - \frac{1}{2}\right)\right], \ 0 < x < 1$$

drawn in Figure 7, which is a bijection from $(0,1)$ to $\mathbb{R}$. We now simply take "pairs" of that bijection to define the function $F : S \to \mathbb{R}^2$ by

$$F(x, y) = (f(x), f(y))$$

Note that this mapping maps the point $(0.5, 0.5) \in S$ into the origin in $\mathbb{R}^2$, and the points in upper right quadrant of S into the first quadrant of $\mathbb{R}^2$, and so on. We leave it to the reader to show this mapping is a bijection. (See Problem 4.)

Summary: We now know that the following sets are equivalent

$$\mathbb{R} \approx (0,1) \approx (0,1) \times (0,1) \approx \mathbb{R}^2$$

Problems

1. **(Visual Correspondences)** Construct visual correspondences between the following sets to show they are equivalent.

   a) $(0,1)$ and $\mathbb{R}$
   b) $(0,\infty)$ and $(-\infty,0)$
   c) $S=\{(x,y): x^2+y^2<1\}$ and $\mathbb{R}^2$

2. **(Irrational Numbers)** Show that the irrational numbers in the interval $[0,1]$ is uncountable.

3. **(Uncountable Set)** Show that the set $\mathbb{R}-\mathbb{Z}$ (i.e. all real numbers that are not integers) is uncountable.

4. **(Bijection from the Unit Square to $\mathbb{R}^2$)** Let

$$S=\{(x,y):0<x<1,\ 0<y<1\}$$

show that the function $F:S\to\mathbb{R}^2$ defined by

$$F(x,y)=(f(x),f(y))$$

where

$$f(x)=\tan\left[\pi\left(x-\frac{1}{2}\right)\right],\ 0<x<1$$

is a bijection from S to $\mathbb{R}^2$. Evaluate the function F at various points inside the unit square S to get a feel for the function.

5. **Cantor-Schroder-Bernstein Theorem** A useful theorem for proving equivalence of two sets, states that if there exists a one-to-one function from A into B (not necessarily onto) and a one-to-one function from B into A (not necessarily onto), then there exists a bijection from one set to the other; i.e. the two sets are equivalent. Use this theorem to prove $(0,1)\approx[0,1]$ by carrying out the following steps.

a) Find a one-to-one function from $(0,1)$ into a subset of $[0,1]$. Hint: trivial.
b) Find a one-to-one function from $[0,1]$ into a subset of $(0,1)$. Hint: trvial.
c) Apply a) and b) to claim $(0,1)\approx[0,1]$.