Periodic Solutions of Nonlinear Boundary-Value Problems of the Second Kind

Abstract: We study the parabolic boundary-value problem of the second kind (which we call Problem A)

\[ \begin{align*}
\text{PDE: } & \quad Lu = f(x,t) \quad (x,t) \in D \times (-\infty, \infty) \\
\text{BC: } & \quad \frac{\partial u}{\partial \eta} + \beta u = g(x,t,u) \quad (x,t) \in \partial D \times (-\infty, \infty)
\end{align*} \]

where

1. \( L = \sum \sum a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} + \sum b_i(x,t) \frac{\partial}{\partial x_i} + c(x,t) - \frac{\partial}{\partial t} \) is a uniformly parabolic operator on \( \overline{D} \times (-\infty, \infty) \)

2. \( D \) is a smooth bounded domain in \( E^n \).

3. The coefficients of \( L \) are continuous and satisfy the following Holder conditions on \( \partial D \times (-\infty, \infty) \)

\[
\begin{align*}
|a_{ij}(x,t) - a_{ij}(x^0,t)| & \leq M |x - x^0|^\alpha \\
|b_i(x,t) - b_i(x^0,t)| & \leq M |x - x^0|^\alpha \\
|c(x,t) - c(x^0,t)| & \leq M |x - x^0|^\alpha
\end{align*}
\]

4. \( \partial D \) belongs to \( C^{1+\alpha} \)

5. \( f \) satisfies the Holder condition on \( D \times (-\infty, \infty) \)

\[
|f(x,t) - f(x^0,t)| \leq |x - x^0|^\alpha
\]

and is uniformly continuous in \( x \) and \( t \) on \( \partial D \times (-\infty, \infty) \).

6. \( \beta, g \) are continuous on \( \partial D \times (-\infty, \infty) \)

Theorem: For the above problem (which ensures the existence of at least one solution) there exists at least one solution \( u = u(x,t) \) periodic in \( t \) with period \( T \) provided:

1. the functions \( a, b, c, f, \beta, g \) are periodic in \( t \) with period \( T \)
2. \( \beta(x,t) \leq \beta_0 < 0, \beta_0 \) a constant on \( (x,t) \in \partial D \times (-\infty, \infty) \)
3. $c(x,t) \leq 0$, $(x,t) \in \partial D \times (-\infty, \infty)$
4. $g = g(x,t,v)$ is continuous on $(x,t) \in \partial D \times (-\infty, \infty)$

**Proof:** The proof uses the Schauder fixed point theorem to show that the operator 
$S : v \rightarrow u$ defined by

$$
Lu = f(x,t) \\
\frac{\partial u}{\partial \eta} + \beta(x,t)u = g(x,t,v)
$$

has a fixed point in the normed space of continuous periodic functions functions on 
$(x,t) \in \bar{D} \times (-\infty, \infty)$.

**Theorem 2:** If $u = u(x,t)$ is a bounded solution of Problem A that satisfies the 
conditions of Theorem 1, and if $g(x,t,v)$ is monotone increasing in $v$ then the solution 
$u = u(x,t)$ is unique.