On $q$-Analogs of Multiple Zeta Values and other Multiple Harmonic Series

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Multiple Harmonic Sums

Example:

$$Z_n^s(s,t,u) = \sum_{n \geq 1} \frac{1}{n^s} \sum_{i=1}^{n} \frac{1}{i^t} \sum_{j=1}^{i} \frac{1}{j^u} \sum_{k=1}^{j} \frac{1}{k^u}$$

$$= \sum_{n \geq i \geq j \geq k \geq 1} i^{-s} j^{-t} k^{-u}$$

- $i \geq j \geq k$ are positive integers
- $n$ may be finite or infinite ($0 \leq n \leq \infty$)
- $s, t, u$ are positive integers ($s, t, u \in \mathbb{Z}^+$)
- $s > 1$ if $n = \infty$

With strict inequalities:

$$Z_n^s(s,t,u) = \sum_{n \geq i \geq j \geq k \geq 1} i^{-s} j^{-t} k^{-u}$$

- $i > j > k$ are positive integers
- $n$ may be finite or infinite ($0 \leq n \leq \infty$)
- $s, t, u$ are positive integers ($s, t, u \in \mathbb{Z}^+$)
- $s > 1$ if $n = \infty$

$$Z_n^{s_1, \ldots, s_m}(s_1, \ldots, s_m) = \sum_{k_1=1}^{n} \frac{1}{k_1^s} \sum_{k_2=1}^{k_1} \frac{1}{k_2^{s_1}} \cdots \sum_{k_m=1}^{k_{m-1}} \frac{1}{k_m^{s_{m-1}}}$$

$$= \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^{m} k_j^{-s_j}$$

- $k_1 \geq k_2 \geq \cdots \geq k_m$ are positive integers
- $n$ may be finite or infinite ($0 \leq n \leq \infty$)
- $s_1, \ldots, s_m$ are positive integers ($s_j \in \mathbb{Z}^+$)
- $s_1 > 1$ if $n = \infty$
\[ Z_n^\leq(s_1, \ldots, s_m) := \sum_{k_1=1}^{n} \sum_{k_2=1}^{k_1-1} \sum_{k_3=1}^{k_2-1} \cdots \sum_{k_m=1}^{k_{m-1}-1} 1 \]

\[ = \sum_{n \geq k_1 > k_2 > \cdots > k_m \geq 1} \prod_{j=1}^{m} k_j^{-s_j} \]

- \(k_1 > k_2 > \cdots > k_m\) are positive integers
- \(n\) may be finite or infinite (\(0 \leq n \leq \infty\))
- \(s_1, \ldots, s_m\) are positive integers (\(\forall j, s_j \in \mathbb{Z}^+\))
- \(s_1 > 1\) if \(n = \infty\)
- \(\zeta(s_1, \ldots, s_m) := Z_n^\leq(s_1, \ldots, s_m)\) is called a multiple zeta value (MZV)

\[ \text{Relationship between } Z_n^\leq \text{ and } Z_n^\geq \]

\[ Z_n^\leq(s) = Z_n^\geq(s), \]

\[ Z_n^\leq(s, t) = Z_n^\geq(s, t) + Z_n^\geq(s + t), \]

\[ Z_n^\leq(s, t, u) = Z_n^\geq(s, t, u) + Z_n^\geq(s + t, u) + Z_n^\geq(s, t + u) + Z_n^\geq(s + t + u). \]

More generally, let \(\tilde{s} = (s_1, s_2, \ldots, s_m)\). Then

\[ Z_n^\leq(\tilde{s}) = \sum Z_n^\geq(\tilde{t}), \]

where the sum is over all \(\tilde{t}\) obtained from \(\tilde{s}\) by replacing any number of commas by plus signs.

\[ \text{Positive } q\text{-Integers} \]

**Definition 1** The \(q\)-analog of \(k \in \mathbb{Z}^+\) is

\[ [k]_q := \sum_{j=0}^{k-1} q^j \]

\[ = 1 + q + q^2 + \cdots + q^{k-1} \]

\[ = \begin{cases} 
1 - q^k, & q \neq 1, \\
1 - q, & q = 1.
\end{cases} \]

Note that \(\lim_{q \to 1} [k]_q = k\).

Can we find reasonable/interesting \(q\)-analogs of multiple harmonic sums?

\[ \text{q-Powers and q-Binomial Coefficients} \]

Let \(0 \leq n \in \mathbb{Z}\) and \(x, y \in \mathbb{R}\).

Define the asymmetric \(q\)-power by

\[ (x + y)_q^n := \prod_{k=0}^{n-1} (x + yq^k). \]

Clearly,

\[ \lim_{q \to 1} (x + y)_q^n = (x + y)^n. \]

The Gaussian binomial coefficient (a.k.a. the \(q\)-binomial coefficient) is defined for \(0 \leq n \in \mathbb{Z}\) and integer \(k\) by

\[ \binom{n}{k} := \frac{(1 - q^n)}{(1 - q^k)(1 - q^{n-k})}, \quad \text{if } 0 \leq k \leq n, \]

\[ 0, \quad \text{otherwise}. \]
The \( q \)-Factorial

If we set
\[
[n]_q := \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q} = \frac{(1 - q^n)_q}{(1 - q)_q},
\]
then evidently
\[
\binom{n}{k} = \frac{[n]_q}{[k]_q[n-k]_q},
\]
and thus
\[
\lim_{q \to 1} \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.
\]

We also have
\[
\binom{n}{k} = \prod_{j=1}^{k} \frac{1 - q^{n-k+j}}{1 - q}, \quad 0 \leq k \leq n,
\]
\[
[n]_q = \sum_{k=0}^{n(n-1)/2} a_k q^k,
\]
\[
a_k = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has } k \text{ inversions}\}.
\]

The \( q \)-Binomial Theorem

Theorem 2 (Finite \( q \)-Binomial Theorem) Let \( x, y \in \mathbb{R} \) and \( 0 \leq n \in \mathbb{Z} \). Then
\[
(x + y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k} x^{n-k} y^k.
\]

Letting \( q \to 1 \), we deduce

Corollary 1 (Classical Binomial Theorem)
\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.
\]

Corollary 2 (\( q \)-Vandermonde Convolution)
\[
\binom{a+b}{n} = \sum_{k=0}^{n} q^{(n-k)(a-k)} \binom{a}{k} \binom{b}{n-k}.
\]

Proof Sketch. Compare coefficients of \( y^n \) in
\[
(1 + y)^a + b = (1 + y)_q^a (1 + yq^b)_q.
\]

The \( q \)-Exponential Function

If \( 0 < q < 1 \), then
\[
\lim_{n \to \infty} \binom{n}{k} = \lim_{n \to \infty} \prod_{j=1}^{k} \frac{1 - q^{n-j}}{1 - q} = \prod_{j=1}^{k} \frac{1}{1 - q^j} = \frac{1}{(1 - q^k)_q}.
\]

Recall the \( q \)-binomial theorem in the form
\[
(1 + x)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k} x^k, \quad 0 \leq n \in \mathbb{Z}.
\]

Thus,
\[
\prod_{j=0}^{\infty} (1 + xq^j) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{1}{[k]_q} \left( \frac{x}{1 - q} \right)^k.
\]

Define the \( q \)-exponential function by
\[
\exp_q(x) := \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q} = (1 + (1 - q)x)_q .
\]

Then
\[
\lim_{q \to 1} \exp_q(x) = e^x, \quad D_q \exp_q(x) = \exp_q(qx).
\]
The $q$-Gamma Function

The $q$-gamma function has infinite product representation

$$\Gamma_q(t) := \frac{(1-q)^{\infty}}{(1-q)x^1(1-q^2)^{\infty}} = (1-q)^{t-1} \prod_{k=1}^{\infty} \frac{1 - q^k}{1 - q^k + k-1}.$$  

For $t > 0$, we have also

$$\Gamma_q(t) := \int_0^\infty x^{t-1} \exp_q(-qx) \, dx.$$  

It can be shown that $\lim_{q \to 1} \Gamma_q(t) = \Gamma(t)$.

Also $\Gamma_q(t+1) = [t]_q \Gamma_q(t)$, $\Gamma_q(1) = 1$, so

$$\Gamma_q(n+1) = [n]_q, \quad 0 \leq n \in \mathbb{Z}.$$  

Askey proved a $q$-analog of the Bohr-Mollerup theorem: If $f(1) = 1$, $f(t+1) = [t]_q f(t)$ for some $0 < q < 1$, and log $f$ is convex for $x > 0$, then $f(x) = \Gamma_q(x)$.

$q$-Harmonic Sums

Suppose we define

$$Z_n[s] := \sum_{n \geq k \geq 1} \frac{1}{k!_q} = (1-q)^s \sum_{k=1}^{n} \frac{1}{(1-q^k)^s}.$$  

If $0 < q < 1$, then $\lim_{k \to \infty} \frac{1}{(1-q^k)^s} = 1$.

Consequently, $Z_\infty[s] = \sum_{k=1}^{\infty} \frac{1}{[k]_q}$ diverges.

Options:

- Restrict $n$ to be finite ($0 \leq n < \infty$)
- Insist that $q > 1$ if $n = \infty$

- Insert convergence factors into the sum

  eg. $\sum_{k=1}^{n} \frac{q^k}{[k]_q}$ or $\sum_{k=1}^{n} \frac{q^{nk}}{[k]_q}$ or $\sum_{k=1}^{n} \frac{q^{(s-1)k}}{[k]_q}$

Multiple $q$-Harmonic Sums

Definition 4 Let $n$, $m$ and $s_1, s_2, \ldots, s_m$ be positive integers. Define

$$Z_n[s_1, \ldots, s_m] := \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q}.$$  

$$A_n[s_1, \ldots, s_m] := \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} (-1)^{k_1+1} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q} \times q^{k_1(k_1+1)/2} \prod_{j=1}^{m} \frac{q^{s_j} - 1}{[k_j]_q}.$$  

Both sums vanish if $n = 0$. If $n > 0$ and $m = 0$, define $Z_n[\cdot] = A_n[\cdot] = 1$.

Abbreviations:

$m \mathcal{C}_[s_j]$ denotes the sequence $s_1, s_2, \ldots, s_m$.

$$\{s\}^m := \prod_{j=1}^{m} \mathcal{C}_[s_j] = s_1 \cdots s_m$$  

(i.e. $m \geq 0$ consecutive copies of $s$).
**Theorem 5** Let \( n, r, a_1, b_1, \ldots, a_r, b_r \in \mathbb{Z}^+ \). Then
\[
Z_n \left[ \mathcal{C}_a \{ \{1 \}^{a_1}, b_1 \} + 1, \{1 \}^{a_r}, b_r \right] = A_n \left[ a_1, \{1 \}^{b_1}, C_a \{ a_j + 1, \{1 \}^{b_j-1} \} \right].
\]

**Example 1** Putting \( r = 2, a_1 = 3, b_1 = 2, a_2 = b_2 = 1 \) gives \( Z_n[1, 1, 3, 1] = A_n[3, 1, 2] \), i.e.
\[
\sum_{n \geq k \geq m \geq 1} q^{i+k+m+p} \frac{n}{[i]_q[k]_q[m]_q[p]_q} = \sum_{n \geq k \geq m \geq 1} (-1)^k q^{k(k-1)/2} \binom{n}{k} \frac{q^m}{[k]_q[m]_q[p]_q}.
\]

**Example 2** Putting \( r = 2, a_1 = a_2 = b_1 = 1, b_2 = 2 \) in Theorem 5 gives \( Z_n[2, 2] = A_n[1, 2, 1] \), i.e.
\[
\sum_{n \geq k \geq m \geq 1} q^{k+m} \frac{n}{[k]_q[m]_q} = \sum_{n \geq k \geq m \geq 1} (-1)^k q^{k(k+1)/2} \binom{n}{k} \frac{q^m}{[k]_q[m]_q}.
\]

**Recall Theorem 5:**
\[
Z_n \left[ \mathcal{C}_a \{ \{1 \}^{a_1}, b_1 \} + 1, \{1 \}^{a_r}, b_r \right] = A_n \left[ a_1, \{1 \}^{b_1}, C_a \{ a_j + 1, \{1 \}^{b_j-1} \} \right].
\]

**Corollary 3** Let \( n, a, b \in \mathbb{Z}^+ \). Then \( Z_n[1, 1, a] = A_n[a, \{1 \}^{b-1}] \).

**Proof.** Put \( r = 1 \) in Theorem 5. \( \square \)

**Example 3** Putting \( b = 1 \) and \( a = m \) yields \( Z_n[1, 1, m] = A_n[m] \), which is Dilcher’s result
\[
\sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^m \frac{q^{k_j}}{[k_j]_q} = \sum_{k_1 \geq \cdots \geq k_m \geq 1} (-1)^{k_1+1} q^{k_1(k_1+1)/2} \binom{n}{k_1} \frac{q^{m-1}}{[k_1]_q[m]_q[k]_q}.
\]

Note the limiting case
\[
\sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \frac{1}{k_j} = \sum_{k_1 \geq \cdots \geq k_m \geq 1} \frac{(-1)^{k_1+1} \binom{n}{k_1}}{k_1}
\]

**Proof of Theorem 5**

By induction, it suffices to establish the base cases
\( A_n[1] = A_n[0] = 1 \) for \( 0 < n \in \mathbb{Z} \) and the following two recurrence relations:

**Proposition 6** Let \( n, m \) and \( s_1, s_2, \ldots, s_m \in \mathbb{Z}^+ \). Then
\[
A_n[s_1, \ldots, s_m] = \sum_{r=1}^n q^r A_{n-r}[s_1-1, s_2, \ldots, s_m].
\]

**Proposition 7** Let \( n, m \) and \( s_2, s_3, \ldots, s_m \in \mathbb{Z}^+ \). Then
\[
A_n[0, s_2, s_3, \ldots, s_m] = A_n[s_2-1, s_3, \ldots, s_m] \frac{[n]_q}{[n]_q}
\]
Base Cases

The base case

\[ A_n[\ ] = 1 \quad \text{for } n > 0 \]

is true by definition.

The other base case, namely

\[ A_n[0] = 1 \quad \text{for } n > 0 \]

is an easy consequence of the \(q\)-binomial theorem:

\[ (x + y)^n_q = \sum_{k=0}^{n} q^{k(k-1)/2} \left[ \begin{array}{c} n \\end{array} \right] q^{-k} x^{n-k} y^k. \]

If \( n > 0 \), Putting \( x = 1 \) and \( y = -1 \) gives

\[ A_n[0] = \sum_{k=1}^{n} (-1)^{k+1} q^{k(k-1)/2} \left[ \begin{array}{c} n \\end{array} \right] q^{-k} \]

\[ = 1 - (1 - 1)^n_q \]

\[ = 1. \]

Definition 9 Let \( n, m, \) and \( s_1, \ldots, s_m \) be non-negative integers. Define

\[ W_n[s_1, \ldots, s_m] := \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^{m} q^{(s_j-1) k_j} \left[ \begin{array}{c} [k_j]_q \end{array} \right]. \]

If \( n = 0 \), the sum is empty and \( W_0[\ ] = 0 \). If \( n > 0 \) and \( m = 0 \), define \( W_n[\ ] := 1 \).

We can now write

\[ A_n[s_1, \ldots, s_m] = \sum_{k=1}^{n} (-1)^{k+1} q^{k(k+1)/2} \left[ \begin{array}{c} n \\end{array} \right] q^{(s_1-1) k} W_k[s_2, \ldots, s_m] \]

\[ = \sum_{k=1}^{n} (-1)^{k+1} q^{k(k+1)/2 + (s_1-1) k} W_k \left[ \begin{array}{c} [k]_q^{s_1} \end{array} \right]. \]

By Lemma 8, the boxed expression is \( \sum_{r=k}^{n} q^{r} \left[ \begin{array}{c} r-1 \\end{array} \right] \).
Multiple Zeta Values

\[ \zeta(s_1, \ldots, s_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j}. \]

The multiple series is absolutely convergent if

\[ \sum_{j=1}^{n} \Re(s_j) > n, \quad n = 1, 2, \ldots, m. \]

Euler (m = 2):

\[ 2 \zeta(s, 1) = s \zeta(s + 1) - \sum_{j=1}^{s-2} \zeta(s - j) \zeta(j + 1), \]

where \( 2 \leq s \in \mathbb{Z} \).

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Period One

For all non-negative integers \( n \),

\[ \zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}, \]

\[ \zeta(\{4\}^n) = \frac{2^{2n+1} \pi^{4n}}{(4n+2)!}, \]

\[ \zeta(\{6\}^n) = \frac{6 \cdot (2\pi)^{6n}}{(6n+3)!}, \]

\[ \zeta(\{8\}^n) = \frac{8 \cdot (2\pi)^{8n}}{(8n+4)!} \times \left\{ \left( 1 + \frac{1}{\sqrt{2}} \right)^{4n+2} + \left( 1 - \frac{1}{\sqrt{2}} \right)^{4n+2} \right\}. \]

More generally, let \( k \in \mathbb{Z}^+ \) and \( \omega := e^{i \pi / k} \). Then

\[ \sum_{n=0}^{\infty} (-1)^n x^{2kn} \zeta(\{2k\}^n) = \prod_{j=0}^{k-1} \frac{\sin(\pi x \omega^j)}{\pi x \omega^j}. \]

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Period Two

For all non-negative integers \( n \),

\[ \zeta(\{3, 1\}^n) = 4^{-n} \zeta(\{4\}^n) = \frac{2 \pi^{4n}}{(4n+2)!}, \]

\[ \zeta(3, \{1, 3\}^n) = 4^{-n} \sum_{k=0}^{n} \zeta(4k + 3) \zeta(\{4\}^{n-k}) \]

\[ = \sum_{k=0}^{n} \frac{2 \pi^{4k}}{(4k+2)!} \left( -\frac{1}{4} \right)^{n-k} \zeta(4n - 4k + 3), \]

\[ \zeta(2, \{1, 3\}^n) = 4^{-n} \sum_{k=0}^{n} (-1)^k \zeta(\{4\}^{n-k}) \left( 4k + 1 \right) \]

\[ \times \zeta(4k + 2) - 4 \sum_{j=1}^{k} \zeta(4j - 1) \zeta(4k - 4j + 3). \]

The study of multiple zeta values touches on

- **Number theory**: algebraic independence
- **Knot theory**: polynomials of Kaufman, Kontsevich
- **Quantum field theory**: Feynman diagrams
- **Algebra**: Quasi-symmetric functions, \( K \)-theory
- **Combinatorics**: shuffles, quad-trees
- **Symbolic computation**: Gröbner bases
- **Numerical computation**: hunting for or disproving the existence of identities
Multiple $q$-Zeta Values

In analogy with

\[ Z_n^{>}[s_1, \ldots, s_m] = \sum_{n > k_1 > \cdots > k_m \geq 1} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}}, \]

define

\[ Z_{\infty}^{>}[s_1, \ldots, s_m] = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}} \]

Then

\[ Z_{\infty}^{>}[s_1, \ldots, s_m] = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}} \]

is a $q$-analog of the multiple zeta value

\[ \zeta(s_1, \ldots, s_m) = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{1}{[k_j]_q^{s_j}}. \]

M. Kaneko investigated analytic properties of the Riemann $q$-zeta function

\[ \zeta[s] := \sum_{n=1}^{\infty} \frac{q^{nk}}{[n]_q^n}, \quad t = s - 1. \]

This suggests we consider

\[ \zeta[s_1, \ldots, s_m] := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}}. \]

Recall we previously defined

\[ Z_{\infty}^{>}[s_1, \ldots, s_m] = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q^{s_j}} \]

Let \( \vec{s} = (s_1, \ldots, s_m) \). Then

\[ \zeta[\vec{s}; q] = q^{[\vec{s}]} Z_{\infty}^{>}[\vec{s}; 1/q], \quad |\vec{s}| := \sum_{j=1}^{m} s_j. \]

Stuffle Multiplication

Example.

\[ \zeta(a) \zeta(b, c) = \sum_{n>0} n^{-a} \sum_{m>k>0} m^{-b} k^{-c} \]

\[ = \sum_{n>m>k>0} n^{-a} m^{-b} k^{-c} \]

\[ + \sum_{n=m>k>0} n^{-a} k^{-b-c} \]

\[ + \sum_{m>n=k>0} m^{-b} n^{-a} k^{-c} \]

\[ + \sum_{m=k>n>0} m^{-b} k^{-a} n^{-c} \]

\[ + \sum_{m=k=n>0} m^{-b} k^{-a} n^{-a} \]

\[ = \zeta(a, b, c) + \zeta(a, b+c) + \zeta(b, a, c) \]

\[ + \zeta(b, a+c) + \zeta(b, c, a). \]
Let $u$ and $v$ be (ordered) lists of positive integers. Then
\[
\zeta(u)\zeta(v) = \sum_{w \in u \ast v} \zeta(w),
\]
where $u \ast v$ is the multiset defined by the recursion
\[
(s, u) \ast (t, v) = (s, u \ast (t, v)) \cup (t, (s, u) \ast v) \cup (s + t, u \ast v),
\]
\[
\varepsilon \ast u = u \ast \varepsilon = u,
\]
where $\varepsilon$ is the empty list.

eg.
\[
a \ast (b, c) = (a, \varepsilon \ast (b, c)) \cup (b, a \ast c) \cup (a + b, \varepsilon \ast c)
\]
\[
= \{(a, (b, c)), (b, a, \varepsilon \ast c), (b, c, a \ast \varepsilon),
(b, a + c, \varepsilon \ast \varepsilon), (a + b, c)\}
\]
\[
= \{(a, b, c), (b, a + c, \varepsilon \ast \varepsilon), (b, c, a)\},
\]
whence
\[
\zeta(a)\zeta(b, c) = \zeta(a, b, c) + \zeta(b, a + c) + \zeta(b, c, a)
\]
\[
+ \zeta(b, a + c) + \zeta(a + b, c).
\]

### Counting Stuffles

Let $f(|u|, |v|)$ denote the number of lists in $u \ast v$.

The recursive decomposition implies that the generating function

\[
F(x, y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n)x^m y^n
\]
satisfies the functional equation

\[
F(x, y) = 1 + xF(x, y) + yF(x, y) + xyF(x, y).
\]

It follows that

\[
F(x, y) = (1 - x - y - xy)^{-1},
\]

\[
f(m, n) = \sum_{k=0}^{m} \binom{m}{k} \binom{n + k}{m} = \sum_{k=0}^{\min(m, n)} \binom{m}{k} \binom{n}{k} 2^k
\]

\[
= \left\lfloor \binom{m}{j} \sum_{j=1}^{m} |b_j| \leq n \right\rfloor.
\]

### Explicit Description

Abbreviate \{1, 2, \ldots, n\} by $\langle n \rangle$.

Define a **stuffle** on $\langle m, n \rangle \in \mathbb{Z}^+ \times \mathbb{Z}^+$ as a pair $(\phi, \psi)$ of order-preserving injective mappings

\[
\phi: \langle m \rangle \to \langle m + n \rangle, \quad \psi: \langle n \rangle \to \langle m + n \rangle
\]
such that the union of their images is equal to $\langle r \rangle$ for some $r$ with $\max(m, n) \leq r \leq m + n$.

Stuffle multiplication can now be written in the form

\[
\zeta(s_1, \ldots, s_m)\zeta(t_1, \ldots, t_n) = \sum_{(\phi, \psi)} \zeta\left(\text{Cat}_{k=1}^{r} \{ s_{\phi^{-1}(k)} + t_{\psi^{-1}(k)} \}\right),
\]

where the sum is over all stuffles $(\phi, \psi)$ on $(m, n)$, and $r = r(\phi, \psi)$ is as above.
\section*{q-Stuffles}

More generally, expanding the product

\[ \zeta[s_1, \ldots, s_m] \zeta[t_1, \ldots, t_n] \]

yields sums of products of terms of the form

\[ q^{(s-1)k+(t-1)j} \frac{[k]^s [j]^t}{[k]^s [j]^t} \]

which, if \( k = j \), reduces to

\[ q^{(s+t-2)k} \frac{[k]^{s+t}}{[k]^{s+t}} = (1 - q)^{s+t} \frac{q^{(s+t-2)k}}{[k]^{s+t-1}} + q^{(s+t-1)k} \frac{[k]^{s+t-1}}{[k]^{s+t-1}} \]

It follows that

\[ \zeta[s_1, \ldots, s_m] \zeta[t_1, \ldots, t_n] = \sum_{(\phi, \psi) \in A} (1-q)^{|A|} \zeta[\text{Cat} \{ s_{\psi^{-1}(k)} + t_{\psi^{-1}(k)} - (k \in A) \}] \]

where the outer sum is over all stuffles \((\phi, \psi)\) on \((m, n)\), the inner sum is over all subsets \(A\) of the intersection of the images of \(\phi\) and \(\psi\), \(r = |\phi(m)| \cup |\psi(n)|\) as before, and the Boolean expression \((k \in A)\) takes the value 1 if true (i.e. \( k \in A \)) and 0 if false (i.e. \( k \notin A \)).

\section*{Period-1 Sums Reduce}

\begin{theorem}
If \(n\) is a positive integer and \(s > 1\), then

\[ n \zeta([s]^n] = \sum_{k=1}^{n} (-1)^{k+1} \zeta([s]^{n-k}) \sum_{j=0}^{k-1} \binom{k-1}{j}(1-q)^j \zeta[k s - j]. \]
\end{theorem}

\begin{example}
With \(n = 2\), we get

\[ 2 \zeta[s, s] = \zeta[s, s] - (\zeta[2s] + (1-q)\zeta[2s - 1]). \]
\end{example}

\begin{corollary}
If \(n\) is a positive integer and \(s > 1\), then

\[ n \zeta([s]^n) = \sum_{k=1}^{n} (-1)^{k+1} \zeta([s]^{n-k}) \zeta(k s). \]
\end{corollary}

\section*{Parity Reduction}

\begin{theorem}
Let \(m \in \mathbb{Z}^+\) and let \(s_1, \ldots, s_m\) be real numbers with \(s_1 > 1\), \(s_m > 1\), and \(s_j \geq 1\) for \(1 < j < m\). Then

\[ \zeta[\text{Cat} s_k] + (-1)^m \zeta[\text{Cat} s_{m-k+1}] \]

can be expressed as a \(\mathbb{Z}[q]\)-linear combination of multiple \(q\)-zeta values of depth less than \(m\).

That is, the coefficients in the linear combination are polynomials in \(q\) with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.
A Double Generating Function

**Theorem 13**

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u^{m+1} v^{n+1} \zeta[m+2, \{1\}^n] = 1 - \exp \left\{ \sum_{k=2}^{\infty} \left( u^k + v^k - (u + v + (1-q)uv)^k \right) \right\} \times \frac{1}{k} \sum_{j=2}^{k} (q-1)^{k-j} \zeta[j].
\]

**Corollary 6** If \( 0 \leq m, n \in \mathbb{Z} \), then \( \zeta[m+2, \{1\}^n] = \zeta[n+2, \{1\}^m] \).

**Corollary 7 (q-Euler)** Let \( 0 \leq m \in \mathbb{Z} \). Then

\[
2\zeta[m+2, 1] = (m+2)\zeta[m+3] + (1-q)(m-1)\zeta[m+2] - \sum_{k=2}^{m+1} \zeta[m+3-k] \zeta[k].
\]

In particular, when \( m = 0 \) we get \( \zeta[2, 1] = \zeta[3] \).

---

The proof of Theorem 13 makes essential use of the basic hypergeometric function

\[
\phi_1 \left[ \frac{q^a q^b}{q^c} \right] x = \sum_{n=0}^{\infty} \frac{(1-q)q^n (1-q^b)^n}{(1-q^c)^n} x^n, \quad |x| < 1.
\]

Routine series manipulations reveal that

\[
\phi_1 \left[ \frac{q^a q^b}{q^c} \right] x = 1 - \phi_1 \left[ \frac{q^a q^b}{q^c} \right] x^q q^a
\]

of Gauss's 2F1 summation formula then gives

\[
\phi_1 \left[ \frac{q^a q^b}{q^c} \right] x = \frac{\Gamma_q(c) \Gamma_q(c-a-b)}{\Gamma_q(c-a) \Gamma_q(c-b)}
\]

**The Simplex Integral**

M. Kontsevich: If \( s_1, \ldots, s_m \in \mathbb{Z}^+ \), then

\[
\zeta(s_1, \ldots, s_m) = \int \prod_{k=1}^{m} \left( \frac{dt_k}{t_k^{s_k+1}} \right) \frac{dt_k^{s_k}}{1-t_k^{s_k}},
\]

where the integral is over the simplex

\[
1 > t_1^{(1)} > \cdots > t_1^{(n)} > \cdots > t_1^{(m)} > \cdots > t_m^{(1)} > \cdots > t_m^{(m)} > 0,
\]

and is abbreviated (D. Broadhurst) by

\[
\int_0^1 \prod_{k=1}^{m} A_k^{s_k-1} B, \quad A = \frac{dt}{t}, \quad B = \frac{dt}{1-t}
\]

\[\text{Theorem 13 is case } w = 0 \text{ of Theorem 14.} \]
\[ \zeta(2,1) = \sum_{n,m > 0} n^{-2} m^{-1} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k + j)^{-2} k^{-1} = \sum_{k=1}^{\infty} k^{-1} \int_{0}^{1} t^{k-1} \int_{0}^{t} u^{k-1} du \, dt \]

\[ = \int_{0}^{1} t^{1} \int_{0}^{t} (1-u)^{-1} \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} u^{k-1} k^{-1} u^{j-1} du \, dt \]

\[ = \int_{0}^{1} t^{1} \int_{0}^{t} (1-u)^{-1} \sum_{k=1}^{\infty} \frac{u^{k-1}}{1-u} \frac{dv}{1-v} \, dt \]

\[ = \int_{0}^{1} AB^{2} \]

---

**The Jackson Integral**

Suppose \( f : (0,b] \to \mathbb{R} \) and \( 0 < x \leq b \).

The Jackson \( q \)-integral of \( f \) on the subinterval \( (0,x] \) is

\[ \int_{0}^{x} f(t) \, dq_{t} := (1-q) \sum_{j=0}^{\infty} xq^{j} f(xq^{j}). \]

If there exists \( 0 < \alpha < 1 \) such that \( |f(t)|^{\alpha} \) is bounded on \((0,b]\), then the integral converges to a function \( F(x) \) on \((0,b]\).

Additionally, \( F \) is a \( q \)-antiderivative of \( f \):

\[ D_{q}F(x) := \frac{F(qx) - F(x)}{(q-1)x} = f(x), \quad 0 < x \leq b \]

Note that

\[ \lim_{q \to 1} D_{q}F(x) = F'(x), \]

and

\[ \lim_{q \to 1} \int_{0}^{x} f(t) \, dq_{t} = \int_{0}^{x} f(t) \, dt. \]

---

**The Jackson Simplex Integral**

Let \( s_{1}, \ldots, s_{m} \) are positive integers. Recall:

\[ \zeta(s_{1}, \ldots, s_{m}) = \prod_{k=1}^{m} \left( \frac{s_{k}^{-1} \delta_{k}^{(k)}}{1 - \delta_{k}^{(k)}} \right) \]

where the integral is over the simplex

\[ 1 > t_{1}^{(1)} > \cdots > t_{1}^{(m)} > \cdots > t_{m}^{(m)} > 0. \]

**Theorem 15**

\[ \zeta(s_{1}, \ldots, s_{m}) = \prod_{k=1}^{m} \left( \frac{\prod_{r=1}^{k-1} \delta_{r}^{(k)}}{1 - \delta_{k}^{(k)}} \right) \]

where

\[ y_{k} := \prod_{j=1}^{k} q_{1}^{-s_{j}}, \]

and the integral is over the same simplex as above.
Generalized Duality

Definition 16 Let \( n \) and \( s_1, \ldots, s_n \) be positive integers with \( s_1 > 1 \). Let \( m \) be a non-negative integer. Define
\[
S(s_1, \ldots, s_n; m) := \sum_{c_1+\cdots+c_n=m} \zeta(s_1 + c_1, \ldots, s_n + c_n).
\]

For positive integers \( a_i \) and \( b_i \), define the dual argument lists
\[
\bar{s} = \Cat_{i=1}^{n} \{ a_i + 1, \{1\}^{b_i-1}\},
\]
\[
\bar{s}' = \Cat_{i=n}^{1} \{ b_i + 1, \{1\}^{a_i-1}\}.
\]

Theorem 17 (Y. Ohno) For any pair of dual argument lists \( \bar{s}, \bar{s}' \) and any non-negative integer \( m \), we have the equality
\[
S(\bar{s}; m) = S(\bar{s}'; m).
\]

---

Generalized \( q \)-Duality

Definition 18 Let \( n \) and \( s_1, \ldots, s_n \) be positive integers with \( s_1 > 1 \). Let \( m \) be a non-negative integer. Define
\[
S(s_1, \ldots, s_n; m) := \sum_{c_1+\cdots+c_n=m} \zeta(s_1 + c_1, \ldots, s_n + c_n).
\]

For positive integers \( a_i \) and \( b_i \), define the dual argument lists
\[
\bar{s} = \Cat_{i=1}^{n} \{ a_i + 1, \{1\}^{b_i-1}\},
\]
\[
\bar{s}' = \Cat_{i=n}^{1} \{ b_i + 1, \{1\}^{a_i-1}\}.
\]

Theorem 19 For any pair of dual argument lists \( \bar{s}, \bar{s}' \) and any non-negative integer \( m \), we have
\[
S(\bar{s}; m) = S(\bar{s}'; m).
\]

---

\( q \)-Sum Formula

Definition 20 Let \( t_1, \ldots, t_n \) be positive integers.
\[
\zeta^*[t_1, \ldots, t_n] := \zeta[t_1 + 1, \Cat_{j=2}^{n} t_j].
\]

Corollary 9 (\( q \)-Sum Formula) For any integers \( 0 < k \leq n \), we have
\[
\sum_{t_1+t_2+\cdots+t_n=k} \zeta^*[t_1, t_2, \ldots, t_n] = \zeta^*[k],
\]
where the sum is over all positive integers \( t_1, \ldots, t_n \) with sum equal to \( k \).

Proof. If we take the dual argument lists in the form \( \bar{s} = (n+1) \) and \( \bar{s}' = (2, \{1\}^{n-1}) \) and put \( m = k - n \), then Theorem 19 states that
\[
\zeta^*[k+1] = \sum_{c_1+\cdots+c_n=k-n} \zeta[2+c_2, \Cat_{j=2}^{n} \{1 + c_j\}]
\]
\[
= \sum_{t_1+\cdots+t_n=k} \zeta[t_1 + 1, \Cat_{j=2}^{n} t_j].
\]

\( \square \)
$q$-Cyclic Sum Formula

**Definition 21** Let $s_j \in \mathbb{Z}^+$ for $1 \leq j \leq n$ and put $\bar{s} = (s_1, \ldots, s_n)$. Let $\sigma$ denote the $n$-cycle $(1 \, 2 \, \cdots \, n)$, and let

$$\mathcal{C}(\bar{s}) := \{(s_{\sigma(1)}, \ldots, s_{\sigma(n)}): 1 \leq j \leq n\}$$

denote the set of cyclic permutations of $\bar{s}$.

Recall the definition

$$\zeta^*[s_1, \ldots, s_n] := \zeta[s_1 + 1, s_2, \ldots, s_n].$$

**Theorem 22** Let $\bar{s}$ and $\bar{s}'$ be dual argument lists. Then

$$\sum_{\bar{t} \in \mathcal{C}(\bar{s})} \zeta^*[\bar{t}] = \sum_{\bar{t} \in \mathcal{C}(\bar{s}')} \zeta^*[\bar{t}].$$

---

**Proof of Generalized $q$-Duality**

Let $\mathfrak{h} = \mathbb{Q} \langle x, y \rangle$ denote the non-commutative polynomial algebra over the rational numbers in two indeterminates $x$ and $y$.

Let $\mathfrak{h}^0$ denote the subalgebra $\mathbb{Q}1 \oplus x\mathfrak{h}y$.

The $\mathbb{Q}$-linear map $\zeta$ is defined on $\mathfrak{h}^0$ by

$$\zeta[1] := \zeta[1] = 1$$

and

$$\zeta \left[ \prod_{i=1}^{s} x^{a_i} \right] = \zeta \left[ \prod_{i=1}^{s} \left( a_i + 1, \{1\}^{b_i} \right) \right],$$

for positive integers $a_i, b_i$ ($1 \leq i \leq s$).

Let $\tau$ be the anti-automorphism of $\mathfrak{h}$ that switches $x$ and $y$.

Then $q$-duality simply says that

$$\zeta[\tau w] = \zeta[w], \quad \forall w \in \mathfrak{h}^0.$$

---

For each $n \in \mathbb{Z}^+$, let $D_n$ be the derivation on $\mathfrak{h}$ that maps $x$ to $0$ and $y$ to $x^ny$.

Let $\theta$ be an indeterminate (formal parameter).

Define

$$\Delta := \sum_{n=1}^{\infty} \frac{D_n}{n} \theta^n \quad \text{and} \quad \sigma := \exp(\Delta).$$

Then

$\Delta$ is a derivation on $\mathfrak{h}[\theta]$, and $\sigma$ is an automorphism of $\mathfrak{h}[\theta]$.

For any word $w \in \mathfrak{h}^0$, define

$$f[w; \theta] := \zeta[\sigma w]$$

and

$$g[w; \theta] := f[\tau w; \theta] = \zeta[\sigma \tau w].$$
Generalized $q$-Duality Reformulated

**Theorem 23** For all $w \in \mathfrak{h}^0$, $f[w; \theta] = g[w; \theta]$, i.e. $\zeta \circ \sigma$ is invariant under ordinary duality $\tau$.

**Theorem 24** Let $a_i, b_i \in \mathbb{Z}^+$ and $\sum_{i=1}^{s}(a_i + b_i) > 2$. Let $\theta' := \theta^q - 1$, and set
\[
\mathbb{I}^m = \{(0,1) \times \cdots \times (0,1)\}.
\]
The generating functions $f$ and $g$ satisfy the difference equation
\[
\sum_{\delta \in \mathbb{I}^{m}} (-\theta)^{\delta \cdot \tau} (1 - q)^{\delta \cdot \epsilon} f \left[ \prod_{i=1}^{s} x^{a_i - \delta_i} y^{b_i - \epsilon_i}; \theta \right] = \sum_{\delta \in \mathbb{I}^{m+1}} (-\theta')^{\delta \cdot \tau - 1} (1 - q)^{\delta \cdot \epsilon} f \left[ \prod_{i=1}^{s} x^{a_i - \delta_i} y^{b_i - \epsilon_i + 1}; \theta' \right].
\]
Here, $\delta$ denotes the ordered tuple whose $i$th component is $1 - \delta_i$, and of course $\delta \cdot \epsilon$ denotes the dot product $\sum_i \delta_i \epsilon_i$. Similarly, $\epsilon$ denotes the ordered tuple whose $i$th component is $1 - \epsilon_i$, and $\delta \cdot \epsilon = \sum_i (1 - \delta_i)(1 - \epsilon_i)$.

One can show that $H(\theta)$ is a meromorphic function of $\theta$ of the form
\[
H(\theta) = \theta^n \sum_{n=1}^{\infty} \frac{h_n}{n! q - \theta q^n},
\]
with at worst simple poles at $\theta = p_n := q^{-n}[n]_q$ for positive integers $n$.

Note that
\[
0 = p_0 < p_1 < p_2 < \cdots < p_{n-1} < p_n < \cdots
\]
and
\[
p'_n = q p_n - 1 = p_{n-1} \quad \text{for all } n \geq 1.
\]
The functional equation
\[
H(\theta) = H(\theta')
\]
thus implies that if $H$ has a pole at $p_n$, then $H$ must also have a pole at $p_{n-1}$.

Since $H$ has no pole at $p_0$, it follows that each $h_n = 0$.

Thus, $H$ vanishes identically and so $f = g$. \qed

**Proof of Theorem 23.** Use induction on the total degree of the word $\prod_{i=1}^{s} x^{a_i} y^{b_i}$.

The base case is clearly satisfied, since the word $xy$ is self-dual.

Now apply Theorem 24 to $f$ and $g$ and subtract the resulting two equations.

The terms whose words have total degree less than $\sum_{i=1}^{s}(a_i + b_i)$ are cancelled by the induction hypothesis.

This leaves us with
\[
(-\theta)^s \left\{ f \left[ \prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta \right] - g \left[ \prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta \right] \right\}
\]
\[
= (-\theta')^s \left\{ f \left[ \prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta' \right] - g \left[ \prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta' \right] \right\}.
\]

Thus, the function
\[
H(\theta) := (-\theta)^s \left\{ f \left[ \prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta \right] - g \left[ \prod_{i=1}^{s} x^{a_i} y^{b_i}; \theta \right] \right\}
\]
satisfies the functional equation $H(\theta) = H(\theta')$, where $\theta' = \theta^q - 1$.

Derivations

**Definition 25** (K. Ihara & M. Kaneko) Define a derivation on $\mathfrak{h}$ for each positive integer $n$ by
\[
\partial_n(x) = x(x+y)^{n-1}, \quad \partial_n(y) = -x(x+y)^{n-1} y.
\]

**Theorem 26** (Ihara & Kaneko) For all positive integers $n$ and words $w \in \mathfrak{h}^0$, $\zeta(\partial_n(w)) = 0$.

**Theorem 27** (q-Analog) For all positive integers $n$ and words $w \in \mathfrak{h}^0$, $\zeta(\partial_n(w)) = 0$.

Theorem 27 is actually equivalent to generalized $q$-duality (Theorem 23).
Proof of Theorem 27

Proof. Let $\sigma = \exp(\Delta)$, $\bar{\sigma} = \tau \sigma$,
\[\Delta = \sum_{n=1}^{\infty} \frac{D_n \eta^n}{n}, \quad \bar{\vartheta} = \sum_{n=1}^{\infty} \frac{\partial_n \eta^n}{n}.\]

Generalized $q$-duality (Theorem 23): $\forall w \in \mathfrak{h}^0$,
$\zeta[\sigma w] = \zeta[\tau \sigma w] = \zeta[\tau \sigma \bar{\sigma} w] \iff (\sigma - \bar{\sigma}) w \in \ker \zeta$.

We show that in fact, $(\sigma - \bar{\sigma}) \mathfrak{h}^0 = \partial \mathfrak{h}^0$.

To prove this, we require the following identity of Ihara and Kaneko.

Proposition 28 $\exp(\bar{\vartheta}) = \bar{\sigma}^{-1}$.

To complete the proof of Theorem 27, observe that since
\[\bar{\vartheta} = \log \left( \bar{\sigma}^{-1} \right) = \log \left( 1 - (\sigma - \bar{\sigma}) \sigma^{-1} \right)\]
\[= - (\sigma - \bar{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n} (\sigma^{-1} \bar{\sigma}^{-1})^{n-1} \sigma^{-1},\]
and
\[\sigma - \bar{\sigma} = (1 - \bar{\sigma}^{-1}) \sigma = (1 - \exp(\bar{\vartheta})) \sigma\]
\[= - \bar{\vartheta} \sum_{n=1}^{\infty} \frac{\eta^{n-1}}{n!} \sigma,\]
we see that
\[\partial \mathfrak{h}^0 \subseteq (\sigma - \bar{\sigma}) \mathfrak{h}^0 \quad \text{and} \quad (\sigma - \bar{\sigma}) \mathfrak{h}^0 \subseteq \partial \mathfrak{h}^0.\]

Thus for the kernel of $\zeta$, we have the equivalences
\[(\sigma - \bar{\sigma}) w \in \ker \zeta \iff \partial w \in \ker \zeta \iff \forall n \in \mathbb{Z}^+, \zeta[\partial_n w] = 0.\]

$\square$