On $q$-Analogs of Multiple Zeta Values and other Multiple Harmonic Series

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September 30, 2004

Multiple Harmonic Sums

$Z_n^\succ(s_1, \ldots, s_m) : = \sum_{k_1 \geq k_2 \geq \cdots \geq k_m \geq 1} \frac{1}{k_1^{s_1} k_2^{s_2} \cdots k_m^{s_m}}$

$= \sum_{n \geq k_1 \geq k_2 \geq \cdots \geq k_m \geq 1} \prod_{j=1}^{m} k_j^{-s_j}$

- $k_1 \geq k_2 \geq \cdots \geq k_m$ are positive integers
- $n$ may be finite or infinite ($0 \leq n \leq \infty$)
- $s_1, \ldots, s_m$ are positive integers ($\forall j, s_j \in \mathbb{Z}^+$)
- $s_1 > 1$ if $n = \infty$

Relationship between $Z_n^\succ$ and $Z_n^\prec$

$Z_n^\succ(s) = Z_n^\prec(s)$,

$Z_n^\succ(s, t) = Z_n^\prec(s, t) + Z_n^\prec(s + t)$,

$Z_n^\succ(s, t, u) = Z_n^\prec(s, t, u) + Z_n^\prec(s + t, u)$

$+ Z_n^\prec(s, t + u) + Z_n^\prec(s + t + u)$. More generally, let $\vec{s} = (s_1, s_2, \ldots, s_m)$. Then

$Z_n^\succ(\vec{s}) = \sum Z_n^\prec(\vec{t})$,

where the sum is over all $\vec{t}$ obtained from $\vec{s}$ by replacing any number of commas by plus signs.
Positive $q$-Integers

**Definition 1** The $q$-analog of $k \in \mathbb{Z}^+$ is

$$[k]_q := \sum_{j=0}^{k-1} q^j$$

$$= 1 + q + q^2 + \cdots + q^{k-1}$$

$$= \begin{cases} 
1 - q^k, & q \neq 1, \\
1, & q = 1.
\end{cases}$$

Note that $\lim_{q \to 1} [k]_q = k$.

Can we find reasonable/interesting $q$-analogs of multiple harmonic sums?

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**The $q$-Factorial**

If we set

$$[n]!_q := \prod_{k=1}^{n} [k]_q = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q},$$

then evidently

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \left[ \frac{[n]!_q}{[k]!_q [n-k]!_q} \right],$$

and thus

$$\lim_{q \to 1} \left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{n!}{k!(n-k)!} = \left( \begin{array}{c} n \\ k \end{array} \right).$$

We also have

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \prod_{j=1}^{k} \frac{1 - q^{n-k+j}}{1 - q^j}, \quad 0 \leq k \leq n.$$
Theorem 3 Let \( n, r, a_1, b_1, \ldots, a_r, b_r \in \mathbb{Z}^+ \). Then
\[
Z_n[\sum_{j=1}^{r-1} \mathcal{C}_n^a \{ \{ 1 \}^{a_j-1}, b_j + 1 \}, \{ 1 \}^{a_r-1}, b_r] = A_n[a_1, \{ 1 \}^{b_1-1}, \mathcal{C}_n^a \{ a_j + 1, \{ 1 \}^{b_j-1} \}].
\]

Example 1 Putting \( r = 2, a_1 = 3, b_1 = 2, a_2 = b_2 = 1 \) gives \( Z_n[1, 1, 3, 1] = A_n[3, 1, 2], \) i.e.
\[
\sum_{n \geq k \geq m \geq p \geq 1} \frac{q^{j+k+m+p}}{[p]_q [q]_q [m]_q [k]_q} = \sum_{n \geq k \geq m \geq p \geq 1} (-1)^{k+1} q^{k(k+1)/2} \frac{q^{2k+p}}{[k]_q [q]_q [m]_q [p]_q}.
\]

Example 2 Putting \( r = 2, a_1 = a_2 = b_1 = 1, b_2 = 2 \) in Theorem 3 gives \( Z_n[2, 2] = A_n[1, 2, 1], \) i.e.
\[
\sum_{n \geq k \geq m \geq p \geq 1} \frac{q^{k+m}}{[k]_q [m]_q} = \sum_{n \geq k \geq m \geq p \geq 1} (-1)^{k+1} q^{k(k+1)/2} \frac{q^m}{[k]_q [q]_q [m]_q [p]_q}.
\]

\( A_n/Z_n \) Duality I

Define an involution on the set \( \mathcal{S} \) of finite sequences of positive integers as follows:

Let \( \alpha \) be the map that sends a sequence in \( \mathcal{S} \) to its sequence of partial sums.

Let \( \beta \) be the involution on strictly increasing sequences in \( \mathcal{S} \) defined by
\[
\beta(a_1, \ldots, a_k) = \{ 1, 2, 3, \ldots, a_k \} \setminus \{ a_1, \ldots, a_{k-1} \}
\]
aranged in increasing order.

Theorem 3 can now be restated as
\[
Z_n[s^\alpha] = A_n[\alpha^{-1} \beta s^\alpha], \quad \forall s^\alpha \in \mathcal{S}, \quad 0 < n \leq \infty.
\]

\( A_n/Z_n \) Duality II

Let \( \mathfrak{h} = \mathbb{Q}(x, y) \) and \( \mathfrak{h}' = \mathbb{Q}1 \oplus \mathfrak{h} y \) and fix \( 0 < n \leq \infty \).

Define \( \mathbb{Q} \)-linear maps \( \bar{A}_n \) and \( \bar{Z}_n \) on \( \mathfrak{h}' \) by
\[
\bar{A}_n[1] := A_n[1] = 1,
\]
\[
\bar{A}_n \left[ \prod_{j=1}^{k} x^{s_j-1} y \right] := A_n[s_1, \ldots, s_k], \quad s_j \in \mathbb{Z}^+,
\]
and similarly for \( \bar{Z}_n \).

Let \( J \) be the automorphism of \( \mathfrak{h} \) that switches \( x \) and \( y \).

Define an involution of \( \mathfrak{h}' \) by
\[
w^* = (Jw)x^{-1}y, \quad \forall w \in \mathfrak{h} y.
\]

Then Theorem 3 can be restated as
\[
\bar{A}_n[w] = \bar{Z}_n[w^*], \quad \forall w \in \mathfrak{h} y.
\]

Recall Theorem 3:
\[
Z_n[\sum_{j=1}^{r-1} \mathcal{C}_n^a \{ \{ 1 \}^{a_j-1}, b_j + 1 \}, \{ 1 \}^{a_r-1}, b_r] = A_n[a_1, \{ 1 \}^{b_1-1}, \mathcal{C}_n^a \{ a_j + 1, \{ 1 \}^{b_j-1} \}].
\]

Corollary 1 Let \( n, a, b \in \mathbb{Z}^+ \). Then
\[
Z_n[\{ 1 \}^{a-1}, b] = A_n[a, \{ 1 \}^{b-1}].
\]

Proof. Put \( r = 1 \) in Theorem 3. \( \square \)

Example 3 Putting \( b = 1 \) and \( a = m \) yields
\[
Z_n[\{ 1 \}^m] = A_n[m],
\]
which is Dilcher’s result
\[
\sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \sum_{j=1}^{m} q^{k_j} \prod_{j=1}^{m} \frac{q^{k_j}}{[k_j]_q} = \sum_{k=1}^{n} (-1)^{k+1} q^{k(k+1)/2} \frac{n}{k} q^{(m-1)k} \frac{1}{[k]_q^m}.
\]
Recall Corollary 1: If \( n, a, b \in \mathbb{Z}^+ \) then
\[
Z_n[\{1\}^{a-1}, b] = A_n[a, \{1\}^{b-1}].
\]

**Example 4** Putting \( a = 1 \) and \( b = m \) yields
\[
Z_n[m] = A_n[\{1\}^m],
\]
i.e.
\[
\sum_{k=1}^{n} \frac{q^k}{[k]^m} = \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \left( -1 \right)^{k_1+1} q^{k_1(k_1+1)/2} \prod_{j=1}^{m} \frac{1}{[k_j]^q},
\]
with limiting case
\[
\sum_{k=1}^{n} \frac{1}{k^m} = \sum_{n \geq k_1 \geq \cdots \geq k_m \geq 1} \left( -1 \right)^{k_1+1} \left( \frac{n}{k_1} \right) \prod_{j=1}^{m} \frac{1}{k_j}.
\]

### Proof of Theorem 3

By induction, it suffices to establish the base cases
\[
A_n[\{1\}^0] = A_n[0] = 1 \text{ for } 0 < n \in \mathbb{Z}
\]
and the following two recurrence relations:

**Proposition 4** Let \( n, m \) and \( s_1, s_2, \ldots, s_m \in \mathbb{Z}^+ \). Then
\[
A_n[s_1, \ldots, s_m] = \sum_{r=1}^{n} q^r A_r[s_1-1, s_2, \ldots, s_m].
\]

**Proposition 5** Let \( n, m \) and \( s_2, s_3, \ldots, s_m \in \mathbb{Z}^+ \). Then
\[
A_n[0, s_2, s_3, \ldots, s_m] = A_n[s_2-1, s_3, \ldots, s_m] \frac{[n]^q}{[n]}.\]

### Multiple Zeta Values

\[
\zeta(s_1, \ldots, s_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j}.
\]

The multiple series is absolutely convergent if
\[
\sum_{j=1}^{n} \Re(s_j) > n, \quad n = 1, 2, \ldots, m.
\]

Euler \((m = 2)\):
\[
2 \zeta(s, 1) = s \zeta(s+1) - \sum_{j=1}^{s-2} \zeta(s-j) \zeta(j+1), \quad 2 \leq s \in \mathbb{Z}.
\]

### Period One

For all non-negative integers \( n, \)
\[
\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n + 1)!},
\]
\[
\zeta(\{4\}^n) = \frac{2^{2n+1} \pi^{4n}}{(4n + 2)!},
\]
\[
\zeta(\{6\}^n) = \frac{6 \cdot (2\pi)^{6n}}{(6n + 3)!},
\]
\[
\zeta(\{8\}^n) = \frac{8 \cdot (2\pi)^{8n}}{(8n + 4)!},
\]
\[
\times \left\{ \left( 1 + \frac{1}{\sqrt{2}} \right)^{4n+2} + \left( 1 - \frac{1}{\sqrt{2}} \right)^{4n+2} \right\}.
\]

More generally, let \( k \in \mathbb{Z}^+ \) and \( \omega := e^{i\pi/k} \). Then
\[
\sum_{n=0}^{\infty} (-1)^n x^{2kn} \zeta(\{2k\}^n) = \prod_{j=0}^{k-1} \frac{\sin(\pi x \omega^j) \prod_{j=0}^{k-1} \sin(\pi x \omega^j)}{\pi x \omega^j}.
\]
Period Two

For all non-negative integers \( n \),

\[
\zeta(\{3,1\}^n) = 4^{-n} \zeta(\{4\}^n) = \frac{2\pi^{4n}}{(4n+2)!}.
\]

\[
\zeta(3, \{1,3\}^n) = 4^{-n} \sum_{k=0}^{n} \zeta(4k+3) \zeta(\{4\}^{n-k})
= \sum_{k=0}^{n} \frac{2\pi^{4k}}{(4k+2)!} \left(\frac{1}{4}\right)^{n-k} \zeta(4n-4k+3),
\]

\[
\zeta(2, \{1,3\}^n) = 4^{-n} \sum_{k=0}^{n} (-1)^k \zeta(\{4\}^{n-k}) \left(4k+1\right)
\times \zeta(4k+2) - 4 \sum_{j=1}^{k} \zeta(4j-1) \zeta(4k-4j+3).
\]

Multiple \( q \)-Zeta Values

Definition 6 Let \( m \) and \( s_1, s_2, \ldots, s_m \) be positive integers and \( 0 < q < 1 \). Define

\[
\zeta[s_1, \ldots, s_m] := \sum_{k_1 > \ldots > k_m > 0} \prod_{j=1}^{m} (q^{s_j-1})_{k_j}^{j}.
\]

Observe that

\[
\lim_{q \to 1} \zeta[s_1, \ldots, s_m] = \zeta(s_1, \ldots, s_m),
\]

where again,

\[
\zeta(s_1, \ldots, s_m) = \sum_{k_1 > \ldots > k_m > 0} \prod_{j=1}^{m} \frac{1}{k_j^{s_j}}.
\]

Also

\[
\zeta[s] \zeta[t] = \zeta[s] + \zeta[t] + \zeta[s + t] + (1 - q)\zeta[s + t - 1].
\]

Period-1 Sums Reduce

Theorem 7 If \( n \) is a positive integer and \( s > 1 \), then

\[
n \zeta(\{s\}^n) = \sum_{k=1}^{n} (-1)^{k+1} \zeta(\{s\}^{n-k}) \sum_{j=0}^{k-1} \binom{k-1}{j} (1-q)^j \zeta[ks-j].
\]

Example 5 With \( n = 2 \), we get

\[
2 \zeta[s, s] = \zeta[s] \zeta[s] - \left(\zeta[2s] + (1 - q)\zeta[2s-1]\right).
\]

Corollary 2 If \( n \) is a positive integer and \( s > 1 \), then

\[
n \zeta(\{s\}^n) = \sum_{k=1}^{n} (-1)^{k+1} \zeta(\{s\}^{n-k}) \zeta(ks).
\]

Let \( S_n \) denote the group of \( n! \) permutations of \( \langle n \rangle = \{1, 2, \ldots, n\} \).

Theorem 8 Let \( n \) be a positive integer, and let \( s_j > 1 \) for \( 1 \leq j \leq n \). Then

\[
\sum_{\sigma \in S_n} \zeta[\text{Cat } s_{\sigma(j)}] = \sum_{P \in \mathcal{P}(n)} (-1)^{n-|P|} \prod_{k=1}^{|P|} (|P_k| - 1)! \times \sum_{\nu_k=0}^{P_k-1} \frac{(|P_k|-1)(1-q)^{\nu_k} \zeta[P_k - \nu_k]}{P_k!},
\]

where the outer sum on the right is over all unordered set partitions \( \mathcal{P} = \{P_1, \ldots, P_m\} \) of \( \langle n \rangle \), \( 1 \leq m = |\mathcal{P}| \leq n \), and \( p_k = \sum_{j \in P_k} s_j \).

Corollary 3 (M. Hoffman)

\[
\sum_{\sigma \in S_n} \zeta[\text{Cat } s_{\sigma(j)}] = \sum_{P \in \mathcal{P}} (-1)^{|P|} \prod_{P \in \mathcal{P}} (|P| - 1)! \zeta\left(\sum_{j \in P} s_j\right).
\]
Parity Reduction

**Theorem 9** Let $m \in \mathbb{Z}^+$ and let $s_1, \ldots, s_m$ be real numbers with $s_1 > 1$, $s_m > 1$, and $s_j \geq 1$ for $1 < j < m$. Then

$$\zeta\left[ \sum_{k=1}^{m} s_k \right] + (-1)^m \zeta\left[ \sum_{k=1}^{m} s_{m-k+1} \right]$$

can be expressed as a $\mathbb{Z}[q]$-linear combination of multiple $q$-zeta values of depth less than $m$.

That is, the coefficients in the linear combination are polynomials in $q$ with integer coefficients.

The proof is a relatively straightforward application of the inclusion-exclusion principle.

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A Double Generating Function

**Theorem 10**

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u^{m+1} v^{n+1} \zeta[m+2, \{1\}^n]$$

$$= 1 - \exp\left\{ \sum_{k=2}^{\infty} \left\{ u^k + v^k - (u + v + (1 - q)uv)^k \right\} \times \frac{1}{k} \sum_{j=2}^{k} (q-1)^{k-j} \zeta[j] \right\}.$$ 

**Corollary 4** If $0 \leq m, n \in \mathbb{Z}$, then

$$\zeta[m+2, \{1\}^n] = \zeta[n+2, \{1\}^m].$$

**Corollary 5 (q-Euler)** Let $0 \leq m \in \mathbb{Z}$. Then

$$2\zeta[m+2, 1] = (m+2)\zeta[m+3] + (1-q)m\zeta[m+2] - \sum_{k=2}^{m+1} \zeta[m+3-k] \zeta[k].$$

$$m = 0 \implies \zeta[2, 1] = \zeta[3].$$

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The Jackson Integral

Suppose $f : (0, b) \to \mathbb{R}$ and $0 < x \leq b$.

The Jackson $q$-integral of $f$ is defined by

$$\int_0^x f(t) \, dq t := (1-q) \sum_{j=0}^{\infty} x^j \, f(x^j).$$

If there exists $0 \leq \alpha < 1$ such that $|f(t)t^\alpha|$ is bounded on $(0, b]$, then the integral converges to a function $F(x)$ on $(0, b]$.

Additionally, $F$ is a $q$-antiderivative of $f$:

$$D_q F(x) := \frac{F(qx) - F(x)}{(q - 1)x} = f(x), \quad 0 < x \leq b.$$ 

Note that

$$\lim_{q \to 1} D_q F(x) = F'(x),$$

and

$$\lim_{q \to 1} \int_0^x f(t) \, dq t = \int_0^x f(t) \, dt.$$ 

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The Jackson Simplex Integral

**Theorem 11** Let $s_1, \ldots, s_m$ are positive integers. Then

$$\zeta[s_1, \ldots, s_m] = \int \prod_{k=1}^{s_m-1} \frac{d q t_k}{t_k^{(k)}} \frac{d q t_{s_k}}{y_k - t_{s_k}},$$

where

$$y_k := \prod_{j=1}^{k} q^{1-s_j},$$

and the integral is over the simplex

$$1 > t_1^{(1)} > \cdots > t_{s_1}^{(1)} > \cdots > t_1^{(m)} > \cdots > t_{s_m}^{(m)} > 0.$$
**Generalized $q$-Duality**

**Definition 12** Let $n$ and $s_1, \ldots, s_n$ be positive integers with $s_1 > 1$. Let $m$ be a non-negative integer. Define

$$S[s_1, \ldots, s_n; m] := \sum_{\substack{c_1, \ldots, c_n \geq 0 \\ c_1 + \cdots + c_n = m}} \zeta[s_1 + c_1, \ldots, s_n + c_n].$$

For positive integers $a_i$ and $b_i$, define the dual argument lists

$$\tilde{s} := \sum_{i=1}^{n} \text{Cat} \{a_i + 1, \{1\}^{b_i-1}\}$$

$$s' \equiv \sum_{i=1}^{n} \text{Cat} \{b_i + 1, \{1\}^{a_i-1}\}. $$

**Theorem 13** For any pair of dual argument lists $\tilde{s}$, $s'$ and any non-negative integer $m$, we have

$$S[\tilde{s}; m] = S[s'; m].$$

**q-Duality**

**Corollary 6** If $\tilde{s}$, $s'$ are dual argument lists, then

$$\zeta[\tilde{s}] = \zeta[s']. $$

In other words, if $a_i, b_i \in \mathbb{Z}^+ (1 \leq i \leq n)$, then

$$\zeta[\text{Cat} \{a_i + 1, \{1\}^{b_i-1}\}] = \zeta[\text{Cat} \{b_i + 1, \{1\}^{a_i-1}\}].$$

**Proof.** Put $m = 0$ in Theorem 13 (generalized $q$-duality). 

**q-Sum Formula**

**Definition 14** Let $t_1, \ldots, t_n$ be positive integers.

$$\zeta^*[t_1, \ldots, t_n] := \zeta[t_1 + 1, \text{Cat} t_j].$$

**Corollary 7 (q-Sum Formula)** For any integers $0 < k \leq n$, we have

$$\sum_{t_1 + t_2 + \cdots + t_n = k} \zeta^*[t_1, t_2, \ldots, t_n] = \zeta^*[k],$$

where the sum is over all positive integers $t_1, \ldots, t_n$ with sum equal to $k$.

**Proof.** If we take the dual argument lists in the form $\tilde{s} = (n + 1)$ and $s' = (2, \{1\}^{n-1})$ and put $m = k - n$, then Theorem 13 states that

$$\zeta[k + 1] = \sum_{\substack{c_1, \ldots, c_n \geq 0 \\ c_1 + \cdots + c_n = k - n}} \zeta[2 + c_2, \text{Cat} \{1 + c_j\}],$$

$$= \sum_{t_1, \ldots, t_n \geq 1} \zeta[t_1 + 1, \text{Cat} t_j].$$

**q-Cyclic Sum Formula**

**Definition 15** Let $s_j \in \mathbb{Z}^+$ for $1 \leq j \leq n$ and put $\tilde{s} = (s_1, \ldots, s_n)$. Let $\sigma$ denote the $n$-cycle $(1 2 \cdots n)$, and let

$$\mathcal{C}(\tilde{s}) := \{(s_{\sigma(1)}, \ldots, s_{\sigma(n)}) : 1 \leq j \leq n\}$$

denote the set of cyclic permutations of $\tilde{s}$.

Recall the definition

$$\zeta^*[s_1, \ldots, s_n] := \zeta[s_1 + 1, s_2, \ldots, s_n].$$

**Theorem 16** Let $\tilde{s}$ and $s'$ be dual argument lists. Then

$$\sum_{\tilde{t} \in \mathcal{C}(\tilde{s})} \zeta^*[\tilde{t}] = \sum_{\tilde{t} \in \mathcal{C}(s')} \zeta^*[\tilde{t}].$$
Proof of Generalized \( q \)-Duality

Let \( \mathfrak{h} = \mathbb{Q}(x, y) \) denote the non-commutative polynomial algebra over the rational numbers in two indeterminates \( x \) and \( y \).

Let \( \mathfrak{h}^0 \) denote the subalgebra \( \mathbb{Q} \oplus x \mathbb{Q} y \).

The \( \mathbb{Q} \)-linear map \( \tilde{\zeta} \) is defined on \( \mathfrak{h}^0 \) by
\[
\tilde{\zeta}[1] := \zeta[1] = 1
\]
and
\[
\tilde{\zeta}\left[ \prod_{i=1}^{s} x^{a_i} y^{b_i} \right] = \zeta\left[ \text{Cat} \left\{ a_i + 1, (1)^{b_i} - 1 \right\} \right],
\]
for positive integers \( a_i, b_i \) \( (1 \leq i \leq s) \).

Let \( \tau \) be the anti-automorphism of \( \mathfrak{h} \) that switches \( x \) and \( y \).

Then \( q \)-duality simply says that
\[
\tilde{\zeta}[\tau w] = \tilde{\zeta}[w], \quad \forall w \in \mathfrak{h}^0.
\]

For each \( n \in \mathbb{Z}^+ \), let \( D_n \) be the derivation on \( \mathfrak{h} \) that maps \( x \mapsto 0 \) and \( y \mapsto x^n y \).

Let \( \theta \) be an indeterminate (formal parameter).

Define
\[
\Delta := \sum_{n=1}^{\infty} \frac{D_n \theta^n}{n} \quad \text{and} \quad \sigma := \exp(\Delta).
\]

Then
\[
\Delta \text{ is a derivation on } \mathfrak{h}[[\theta]], \quad \text{and} \quad \sigma \text{ is an automorphism of } \mathfrak{h}[[\theta]].
\]

Theorem 13 (generalized \( q \)-duality) can now be reformulated as
\[
\tilde{\zeta}[\sigma w] = \tilde{\zeta}[\sigma \tau w], \quad \forall w \in \mathfrak{h}^0.
\]

In other words, \( \tilde{\zeta} \circ \sigma \) is invariant under ordinary duality \( \tau \).

Derivations

Definition 17 (K. Ihara & M. Kaneko) Define a derivation on \( \mathfrak{h} \) for each positive integer \( n \) by
\[
\partial_n(x) = x(x+y)^{n-1}, \quad \partial_n(y) = -x(x+y)^{n-1}y.
\]

Theorem 18 (Ihara & Kaneko) For all positive integers \( n \) and words \( w \in \mathfrak{h}^0 \)
\[
\tilde{\zeta}(\partial_n(w)) = 0.
\]

Theorem 19 (\( q \)-Analog) For all positive integers \( n \) and words \( w \in \mathfrak{h}^0 \),
\[
\tilde{\zeta}[\partial_n(w)] = 0.
\]

Theorem 19 is actually equivalent to generalized \( q \)-duality (Theorem 13).

Proof of Theorem 19

Proof. Let \( \sigma = \exp(\Delta), \bar{\sigma} = \tau \sigma \tau \),
\[
\Delta = \sum_{n=1}^{\infty} \frac{D_n \theta^n}{n} \quad \text{and} \quad \sigma = \sum_{n=1}^{\infty} \frac{\partial_n \theta^n}{n}.
\]

Generalized \( q \)-duality (Theorem 13): \( \forall w \in \mathfrak{h}^0 \),
\[
\tilde{\zeta}[\sigma w] = \tilde{\zeta}[\sigma \tau w] = \tilde{\zeta}[\tau \sigma w] \iff (\sigma - \bar{\sigma})w \in \ker \tilde{\zeta}.
\]

We show that in fact, \( (\sigma - \bar{\sigma})\mathfrak{h}^0 = \partial \mathfrak{h}^0 \).

To prove this, we require the following identity of Ihara and Kaneko.

Proposition 20 \( \exp(\bar{\sigma}) = \bar{\sigma} \sigma^{-1} \).
To complete the proof of Theorem 19, observe that since
\[ \vartheta = \log(\sigma \sigma^{-1}) = \log(1 - (\sigma - \tilde{\sigma})\sigma^{-1}) \]
\[ = - (\sigma - \tilde{\sigma}) \sum_{n=1}^{\infty} \frac{1}{n}((\sigma - \tilde{\sigma})\sigma^{-1})^{n-1} \sigma^{-1}, \]
and
\[ \sigma - \tilde{\sigma} = (1 - \tilde{\sigma})\sigma = (1 - \exp(\vartheta))\sigma \]
\[ = -\vartheta \sum_{n=1}^{\infty} \frac{\vartheta^{n-1}}{n!} \sigma, \]
we see that
\[ \partial h^0 \subseteq (\sigma - \sigma)h^0 \quad \text{and} \quad (\sigma - \sigma)h^0 \subseteq \partial h^0. \]
Thus for the kernel of \( \hat{\zeta} \), we have the equivalences
\[ (\sigma - \sigma)w \in \ker \hat{\zeta} \quad \iff \quad \partial w \in \ker \hat{\zeta} \]
\[ \iff \quad \forall n \in \mathbb{Z}^+, \, \hat{\zeta}[\partial_n w] = 0. \]
\[ \square \]