ON TWO FUNDAMENTAL IDENTITIES FOR EULER SUMS

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Abstract. We give diverse proofs of the fundamental identities $\zeta(2,1) = \zeta(3) = 8\zeta(2,1)$. We also discuss various generalizations for multiple harmonic (Euler) sums and some connections, thereby illustrating the wide variety of techniques fruitfully used to study such sums.

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1. Introduction

For positive integers \( s_1, \ldots, s_m \) and signs \( \sigma_j = \pm 1 \), consider \( \zeta \) the \( m \)-fold Euler sum

\[
\zeta(s_1, \ldots, s_m; \sigma_1, \ldots, \sigma_m) := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{\sigma_j k_j^{s_j}}{k_j^{s_j}}.
\]

As is now customary, we combine strings of exponents and signs by replacing \( s_j \) by \( \overline{s_j} \) in the argument list if and only if \( \sigma_j = -1 \), and denote \( n \) repetitions of a substring \( S \) by \( \{S\}^n \). Thus, for example, \( \zeta(\overline{1}) = -\log 2 \), \( \zeta(\{2\}^3) = \zeta(2, 2, 2) = \pi^6/7! \) and

\[
\zeta(s_1, \ldots, s_m) = \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} k_j^{-s_j}.
\]

The identity

\[
\zeta(2, 1) = \zeta(3)
\]
goes back to Euler \[34\] \[35, p. 228\] and has since been repeatedly rediscovered \[24, 25, 36, 45\]. The more general formula

\[
2\zeta(m, 1) = m\zeta(m + 1) - \sum_{j=1}^{m-2} \zeta(j + 1)\zeta(m - j), \quad 2 \leq m \in \mathbb{Z}
\]  

(1.3)
is also due to Euler \[34\] \[35, p. 266\]. Nielsen \[50, p. 229\] \[51, p. 198\] \[52, pp. 47–49\] developed a method for obtaining (1.3) and related results based on partial fractions. Formula (1.3) has also been rediscovered a number of times \[63, 57, 55, 38, 17, 60\]. Crandall and Buhler \[32\] deduced (1.3) from their general infinite series formula which expresses \(\zeta(s, t)\) for real \(s > 1\) and \(t \geq 1\) in terms of Riemann zeta values.

Study of the the multiple zeta function (1.1) led to the discovery of a new generalization of (1.2), involving nested sums of arbitrary depth:

\[
\zeta(\{2, 1\}^n) = \zeta(\{3\}^n), \quad n \in \mathbb{Z}^+.
\]  

(1.4)

Although numerous proofs of (1.2) and (1.3) are known (we give several in the sequel), the only proof of (1.4) of which we are aware involves making a simple change of variable in a multiple iterated integral (see \[9, 10, 14\] and (5.11) below).

An alternating version of (1.2) is

\[
8\zeta(\overline{2}, 1) = \zeta(3),
\]  

(1.5)

which has also resurfaced from time to time \[52\ p. 50\] \[58, (2.12)\] \[28\ p. 267\] and hints at the generalization

\[
8^n\zeta(\overline{2}, 1)^n = \zeta(\{3\}^n), \quad n \in \mathbb{Z}^+,
\]  

(1.6)

originally conjectured in \[9\], and which still remains open—despite abundant evidence \[7\]. The purpose of this paper is to illustrate some of the techniques used to study Euler sums by focusing on the identities (1.2), (1.5) and various generalizations. For some of the broader issues relating to Euler sums, we refer the reader to the survey articles \[14, 29, 61, 62, 64\]. Computational issues are discussed in \[31\] and to an extent in \[10\].

Notation and Terminology. For positive integer \(N\), denote the \(N\)th partial sum of the harmonic series by \(H_N := \sum_{n=1}^{N} 1/n\). We also use \(\psi = \Gamma'/\Gamma\) to denote the logarithmic derivative of the Euler gamma function (also referred to as the digamma function), and recall the identity \(\psi(N + 1) + \gamma = H_N\), where \(\gamma = 0.5772156649\ldots\) is Euler’s constant. As usual, the Kronecker \(\delta_{m,n}\) is 1 if \(m = n\) and 0 otherwise.

We organize the proofs by technique, although clearly this is somewhat arbitrary as many proofs fit well within more than one category. In some of the later sections the proofs become more schematic. We invite readers to add selections to our collection.
2. Telescoping and Partial Fractions

For a quick proof of (1.2), consider

\[ S := \sum_{n,k>0} \frac{1}{nk(n+k)} = \sum_{n,k>0} \frac{1}{n^2} \left( \frac{1}{k} - \frac{1}{n+k} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{k} = \zeta(3) + \zeta(2,1). \]

On the other hand,

\[ S = \sum_{n,k>0} \left( \frac{1}{n} + \frac{1}{k} \right) \frac{1}{(n+k)^2} = \sum_{n,k>0} \frac{1}{n(n+k)} + \sum_{n,k>0} \frac{1}{k(n+k)^2} = 2\zeta(2,1), \]

by symmetry. \( \square \)

The above argument goes back at least to Steinberg [45]. See also [46].

For (1.5), first consider

\[ \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{(-1)^k}{k} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \left( \frac{(-1)^k}{n+k} \right) \]

\[ = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{n+k - (-1)^nk}{k(n+k)} \]

\[ = \sum_{n,k>0} \frac{(-1)^{n+k}}{nk(n+k)} + \sum_{n,k>0} \frac{(-1)^{n+k}}{n^2(n+k)} - \sum_{n,k>0} \frac{(-1)^k}{n^2(n+k)} \]

\[ = \sum_{n,k>0} \frac{1}{n(n+k)^2} + \sum_{n,k>0} \frac{(-1)^{n+k}}{k(n+k)^2} + \zeta(1,2) - \sum_{n,k>0} \frac{(-1)^n(-1)^{n+k}}{n^2(n+k)} \]

\[ = 2\zeta(2,1) + \zeta(1,2) - \zeta(1,2). \]

Similarly,

\[ \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{(-1)^k}{k} = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} \left( \frac{(-1)^k}{n+k} \right) \]

\[ = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{\infty} (-1)^k \frac{n+k - (-1)^nk}{k(n+k)} \]

\[ = \sum_{n,k>0} \frac{(-1)^k}{nk(n+k)} + \sum_{n,k>0} \frac{(-1)^k}{n^2(n+k)} - \sum_{n,k>0} \frac{(-1)^{n+k}}{n^2(n+k)} \]

\[ = 2\zeta(2,1) + \zeta(1,2) - \zeta(1,2). \]

(2.1)
EULER SUMS

\[ E = \sum_{n,k>0} \left( \frac{1}{n} + \frac{1}{k} \right) (n+k)^2 \left( -1 \right)^k + \sum_{n,k>0} \left( -1 \right)^n (-1)^{n+k} \frac{1}{n(n+k)^2} - \zeta(2,2) \]

\[ = \sum_{n,k>0} \frac{(-1)^n (-1)^n}{n(n+k)^2} + \sum_{n,k>0} \frac{(-1)^k}{k(n+k)^2} + \zeta(2,2) - \zeta(2,2) \]

\[ = \zeta(2,2) + \zeta(2,2) + \zeta(2,2) - \zeta(2,2). \] (2.2)

Adding equations (2.1) and (2.2) now gives

\[ 2\zeta(2,1) = \zeta(3) + \zeta(3), \] (2.3)

i.e.

\[ 8\zeta(2,1) = 4 \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n^3} = 4 \sum_{m=1}^{\infty} \frac{2}{(2m)^3} = \zeta(3), \]

which is (1.5).

3. Finite Series Transformations

For any positive integer \( N \), we have

\[ \sum_{n=1}^{N} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{N-k+1} \] (3.1)

by induction. Alternatively, consider

\[ T := \sum_{n,k=1 \atop k \neq n}^{N} \frac{1}{nk(k-n)} = \sum_{n,k=1 \atop k \neq n}^{N} \left( \frac{1}{n} - \frac{1}{k} \right) \frac{1}{(k-n)^2} = 0. \]

On the other hand,

\[ T = \sum_{n,k=1 \atop k \neq n}^{N} \frac{1}{n^2} \left( \frac{1}{k-n} - \frac{1}{k} \right) \]

\[ = \sum_{n=1}^{N} \frac{1}{n^2} \left( \sum_{k=1}^{n-1} \frac{1}{k-n} + \sum_{k=n+1}^{n} \frac{1}{k-n} - \sum_{k=1}^{N} \frac{1}{k+n} + \sum_{k=1}^{N} \frac{1}{k} \right) \]

\[ = \sum_{n=1}^{N} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k-n} + \sum_{n=1}^{N} \frac{1}{n^2} \left( \sum_{k=n+1}^{N} \frac{1}{k-n} - \sum_{k=1}^{N} \frac{1}{k} \right). \]
Since \( T = 0 \), this implies that
\[
\sum_{n=1}^{N} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n-1} \frac{1}{k} = \sum_{n=1}^{N} \frac{1}{n^2} \left( \sum_{k=1}^{N} \frac{1}{k} - \sum_{k=1}^{N-n} \frac{1}{k} \right) = \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{N-k+1},
\]
which is (3.1). But the right hand side satisfies
\[
\frac{H_N}{N} = \sum_{n=1}^{N} \frac{1}{n^2} \cdot \frac{n}{N} \leq \sum_{n=1}^{N} \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{N-k+1}
\leq \sum_{n=1}^{N} \frac{1}{n^2} \cdot \frac{n}{N-n+1} = \frac{1}{N+1} \sum_{n=1}^{N} \left( \frac{1}{n} + \frac{1}{N-n+1} \right) = \frac{2H_N}{N+1}.
\]
Letting \( N \) grow without bound now gives (1.2), since \( \lim_{N \to \infty} \frac{H_N}{N} = 0 \). \( \square \)

4. Geometric Series

4.1. Convolution of Geometric Series. The following argument is suggested in [63]. A closely related derivation, in which our explicit consideration of the error term is suppressed by taking \( N \) infinite, appears in [17]. Let \( 2 \leq m \in \mathbb{Z} \), and consider
\[
\sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j) = \lim_{N \to \infty} \sum_{n=1}^{N} \sum_{k=1}^{N} \sum_{j=1}^{m-2} \frac{1}{n^j} \frac{1}{k^{m-j}}
= \lim_{N \to \infty} \left\{ \sum_{n,k=1}^{N} \left( \frac{1}{n^{m-1}(k-n)} - \frac{1}{n(k-n)k^{m-1}} \right) + \sum_{n=1}^{N} \frac{m-2}{n^{m+1}} \right\}
= (m-2)\zeta(m+1) + 2 \lim_{N \to \infty} \sum_{n,k=1}^{N} \frac{1}{n^{m-1}k(k-n)}.
\]
Thus, we find that
\[
(m-2)\zeta(m+1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j)
= 2 \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^m} \sum_{k=1}^{N} \frac{1}{k} \sum_{k \neq n} \left( \frac{1}{k} - \frac{1}{k-n} \right)
\]
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\[ = 2 \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^m} \left\{ \sum_{k=1}^{n-1} \frac{1}{k} - \frac{1}{n} + \sum_{k=1}^{n} \frac{1}{N - k + 1} \right\} \]

\[ = 2\zeta(m, 1) - 2\zeta(m + 1) + 2 \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^m} \sum_{k=1}^{n} \frac{1}{N - k + 1}, \]

and hence

\[ 2\zeta(m, 1) = m\zeta(m + 1) - \sum_{j=1}^{m-2} \zeta(j+1)\zeta(m-j) - 2 \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^m} \sum_{k=1}^{n} \frac{1}{N - k + 1}. \]

But, in light of

\[ \sum_{n=1}^{N} \frac{1}{n^m} \sum_{k=1}^{n} \frac{1}{N - k + 1} \leq \sum_{n=1}^{N} \frac{1}{n^m} \cdot \frac{n}{N - k + 1} \leq \frac{1}{N + 1} \sum_{n=1}^{N} \left( \frac{1}{N - n + 1} - \frac{1}{n} \right) = \frac{2H_N}{N + 1}, \]

the identity \((1.3)\) now follows.

\[ \square \]

4.2. A Sum Formula. Equation (1.2) is the case \(n = 3\) of the following result. See [23].

**Theorem 1.** If \(3 \leq n \in \mathbb{Z}\) then

\[ \zeta(n) = \sum_{j=1}^{n-2} \zeta(n-j, j). \quad (4.1) \]

We discuss a generalization \((9.4)\) of the sum formula \((4.1)\) to arbitrary depth in \(\S 9.2\).

**Proof.** Summing the geometric series on the right hand side gives

\[ \sum_{j=1}^{n-2} \sum_{h=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{h^j(h+m)^{n-j}} = \sum_{h, m=1}^{\infty} \frac{1}{h^{n-2}m(h+m)^{n-1}} \]

\[ = \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{m=1}^{\infty} \left( \frac{1}{m} - \frac{1}{h+m} \right) - \zeta(n-1, 1) \]

\[ = \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{k=1}^{h} \frac{1}{k} - \zeta(n-1, 1) \]

\[ = \sum_{h=1}^{\infty} \frac{1}{h^n} + \sum_{h=1}^{\infty} \frac{1}{h^{n-1}} \sum_{k=1}^{n-1} \frac{1}{k} - \zeta(n-1, 1) \]

\[ = \zeta(n). \]

\[ \square \]
4.3. A q-Analog. The following argument is based on an idea of Zudilin [22]. We begin with the finite geometric series identity

\[
\frac{uv}{(1-u)(1-uv)^s} + \frac{uv^2}{(1-v)(1-uv)^s} = \frac{uv}{(1-u)(1-v)^s} - \sum_{j=1}^{s-1} \frac{uv^2}{(1-v)^{j+1}(1-uv)^{s-j}},
\]

valid for all positive integers \(s\) and real \(u, v\) with \(u \neq 1, uv \neq 1\). We now assume \(s > 1\), \(q\) is real and \(0 < q < 1\). Put \(u = q^m, v = q^n\) and sum over all positive integers \(m\) and \(n\). Thus,

\[
\sum_{m,n>0} \frac{q^{m+n}}{(1-q^m)(1-q^{m+n})^s} + \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)(1-q^{m+n})^s} = \sum_{m,n>0} \frac{q^{m+n}}{(1-q^n)(1-q^{m+n})^s} - \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)(1-q^{m+n})^s}
\]

\[
- \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}
\]

\[
= \sum_{m,n>0} \frac{q^n}{(1-q^n)^s} \left[ \frac{q^m}{1-q^m} - \frac{q^{m+n}}{1-q^{m+n}} \right] - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}
\]

\[
= \sum_{n>0} \frac{q^n}{(1-q^n)^s} \sum_{m=1}^{n} \frac{q^m}{1-q^m} - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}
\]

\[
= \sum_{n>0} \frac{q^{2n}}{(1-q^n)^{s+1}} + \sum_{n,m>0} \frac{q^{n+m}}{(1-q^n)^s(1-q^m)} - \sum_{j=1}^{s-2} \sum_{m,n>0} \frac{q^{m+2n}}{(1-q^n)^{j+1}(1-q^{m+n})^{s-j}}
\]

Cancelling the second double sum on the left with the corresponding double sum on the right and replacing \(m + n\) by \(k\) in the remaining sums now yields

\[
\sum_{k>m>0} \frac{q^k}{(1-q^k)(1-q^m)^s} = \sum_{n>0} \frac{q^{2n}}{(1-q^n)^{s+1}} - \sum_{j=1}^{s-2} \sum_{k>m>0} \frac{q^{k+m}}{(1-q^m)^{j+1}(1-q^k)^{s-j}},
\]

or equivalently, that

\[
\sum_{k>0} \frac{q^{2k}}{(1-q^k)^{s+1}} = \sum_{k>m>0} \frac{q^k}{(1-q^k)(1-q^m)^s} + \sum_{j=1}^{s-2} \sum_{k>m>0} \frac{q^{k+m}}{(1-q^k)^{s-j}(1-q^m)^{j+1}}. 
\]  (4.2)

Multiplying (4.2) through by \((1-q)^{s+1}\) and letting \(q \to 1\) gives

\[
\zeta(s + 1) = \zeta(s, 1) + \sum_{j=1}^{s-2} \zeta(s - j, j + 1),
\]
which is just a restatement of (4.1). Taking $s = 2$ gives (1.2) again.

As in [19], define the $q$-analog of a non-negative integer $n$ by

$$[n]_q := \sum_{k=0}^{n-1} q^k = \frac{1 - q^n}{1 - q},$$

and the multiple $q$-zeta function

$$\zeta[s_1, \ldots, s_m] := \sum_{k_1 > \cdots > k_m > 0} \prod_{j=1}^{m} \frac{q^{(s_j-1)k_j}}{[k_j]_q^{s_j}},$$

(4.3)

where $s_1, s_2, \ldots, s_m$ are real numbers with $s_1 > 1$ and $s_j \geq 1$ for $2 \leq j \leq m$. Then multiplying (4.2) by $(1 - q)^{s+1}$ and then setting $s = 2$ gives $\zeta[2, 1] = \zeta[3]$, which is a $q$-analog of (1.2). That is, the latter may be obtained from the former by letting $q \to 1^-$. On the other hand, $s = 3$ in (4.2) gives


$$\zeta[3, 1] = \zeta[4] - \zeta[2, 2] = \frac{3}{2} \zeta[4] - \frac{1}{2} (\zeta[2])^2 + \frac{1}{2} (1 - q)\zeta[3],$$

which is a $q$-analog of the evaluation [9, 10, 11, 13, 14, 15, 16]

$$\zeta(3, 1) = \frac{\pi^4}{360}.$$

Additional material concerning $q$-analogs of multiple harmonic sums and multiple zeta values can be found in [19, 20, 21].

5. Integral Representations

5.1. Single Integrals I. We use the fact that

$$\int_0^1 u^{k-1}(- \log u) \, du = \frac{1}{k^2}, \quad k > 0.$$

(5.1)

Thus

$$\sum_{k>n>1} \frac{1}{k^2n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k>n} \int_0^1 u^{k-1}(- \log u) \, du$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 (\log u) \sum_{k>n} u^{k-1} \, du$$
\[
\sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{1} (-\log u) \frac{u^n}{1-u} \, du
= - \int_{0}^{1} \log u \sum_{n=1}^{\infty} \frac{u^n}{n} \, du
= \int_{0}^{1} (-\log u)(1-u)^{-1} \log(1-u)^{-1} \, du. \tag{5.2}
\]

The interchanges of summation and integration are in each case justified by Lebesgue’s monotone convergence theorem. After making the change of variable \( t = 1-u \), we obtain
\[
\sum_{k>n>0} \frac{1}{k^n} = \int_{0}^{1} \log(1-t)^{-1}(-\log t) \frac{dt}{t} = \int_{0}^{1} (-\log t) \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \, dt. \tag{5.3}
\]

Again, since all terms of the series are positive, Lebesgue’s monotone convergence theorem permits us to interchange the order of summation and integration. Thus, invoking (5.1) again, we obtain
\[
\sum_{k>n>0} \frac{1}{k^2n} = \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{1} (-\log t) t^{n-1} \, dt = \sum_{n=1}^{\infty} \frac{1}{n^3},
\]
which is (1.2). \( \square \)

5.2. Single Integrals II. The Laplace transform
\[
\int_{0}^{1} x^{r-1}(-\log x)^{\sigma} \, dx = \int_{0}^{\infty} e^{-ru} u^{\sigma} \, du = \frac{\Gamma(\sigma+1)}{r^{\sigma+1}}, \quad r > 0, \quad \sigma > -1, \tag{5.4}
\]
generalizes (5.1) and yields the representation
\[
\zeta(m+1) = \frac{1}{m!} \sum_{r=1}^{\infty} \frac{\Gamma(m+1)}{r^{m+1}} = \frac{1}{m!} \sum_{r=1}^{\infty} \int_{0}^{1} x^{r-1}(-\log x)^{m} \, dx = \frac{(-1)^m}{m!} \int_{0}^{1} \log^m x \frac{dx}{1-x}.
\]
The interchange of summation and integration is valid if \( m > 0 \). The change of variable \( x \mapsto 1-x \) now yields
\[
\zeta(m+1) = \frac{(-1)^m}{m!} \int_{0}^{1} \log^m(1-x) \frac{dx}{x}, \quad 1 \leq m \in \mathbb{Z}. \tag{5.5}
\]

In [36], equation (5.4) in conjunction with clever use of change of variable and integration by parts, is used to prove the identity
\[
k!\zeta(k+2) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} \sum_{p=1+n_1+n_2+\cdots+n_k}^{\infty} \frac{1}{p^k}, \quad 0 \leq k \in \mathbb{Z}. \tag{5.6}
\]
The case $k = 1$ of (5.6) is precisely (1.2). We give here a slightly simpler proof of (5.6), dispensing with the integration by parts.

From (5.4),

$$k! \zeta(k + 2) = \sum_{r=1}^{\infty} \frac{1}{r} \cdot \frac{\Gamma(k + 1)}{r^{k+1}} = \sum_{r=1}^{\infty} \frac{1}{r} \int_{0}^{1} x^{r-1}(-\log x)^{k} \, dx$$

$$= \int_{0}^{1} (-\log x)^{k} \log(1 - x)^{-1} \frac{dx}{x}$$

$$= \int_{0}^{1} \log^{k}(1 - x)^{-1}(-\log x) \frac{dx}{1 - x}$$

$$= \sum_{n_{1}=1}^{\infty} \sum_{n_{2}=1}^{\infty} \cdots \sum_{n_{k}=1}^{\infty} \frac{1}{n_{1}n_{2}\cdots n_{k}} \int_{0}^{1} \frac{x^{n_{1}+n_{2}+\cdots+n_{k}}}{1 - x} (-\log x) \, dx$$

$$= \sum_{n_{1}=1}^{\infty} \sum_{n_{2}=1}^{\infty} \cdots \sum_{n_{k}=1}^{\infty} \frac{1}{n_{1}n_{2}\cdots n_{k}} \sum_{p>n_{1}+n_{2}+\cdots+n_{k}} \int_{0}^{1} x^{p-1}(-\log x) \, dx$$

$$= \sum_{n_{1}=1}^{\infty} \sum_{n_{2}=1}^{\infty} \cdots \sum_{n_{k}=1}^{\infty} \frac{1}{n_{1}n_{2}\cdots n_{k}} \sum_{p>n_{1}+n_{2}+\cdots+n_{k}} \frac{1}{p^{2}}.$$ 

\[\square\]

5.3. **Double Integrals I.** Write

$$\zeta(2, 1) = \sum_{k, m > 0} \frac{1}{k(m + k)^{2}} = \int_{0}^{1} \int_{0}^{1} \sum_{k > 0} \frac{(xy)^{k}}{k} \sum_{m > 0} (xy)^{m-1} \, dx \, dy$$

$$= -\int_{0}^{1} \int_{0}^{1} \frac{\log(1 - xy)}{1 - xy} \, dx \, dy.$$ 

Now make the change of variable $u = xy, v = x/y$ with Jacobian $1/(2v)$, obtaining

$$\zeta(2, 1) = -\frac{1}{2} \int_{0}^{1} \frac{\log(1 - u)}{1 - u} \int_{u}^{1/u} \frac{dv}{v} \, du = \int_{0}^{1} \frac{\log u \log(1 - u)}{1 - u} \, du,$$

which is (5.2). Now continue as in 5.1. \[\square\]

5.4. **Double Integrals II.** The following is reconstructed from a phone conversation with Krishna Alladi. Let $\varepsilon > 0$. By expanding the integrand as a geometric series, one sees that

$$\sum_{n=1}^{\infty} \frac{1}{(n + \varepsilon)^{2}} = \int_{0}^{1} \int_{0}^{1} \frac{(xy)^{\varepsilon}}{1 - xy} \, dx \, dy.$$
Differentiating with respect to $\varepsilon$ and then letting $\varepsilon = 0$ gives
\[
\zeta(3) = -\frac{1}{2} \int_0^1 \int_0^1 \log(xy) \frac{dx}{1-xy} \frac{dy}{1-xy} = -\frac{1}{2} \int_0^1 \int_0^1 \log x + \log y \frac{dx}{1-xy} \frac{dy}{1-xy} = -\int_0^1 \int_0^1 \frac{\log x}{1-xy} dx
\]
by symmetry. Now integrate with respect to $y$ to get
\[
\zeta(3) = \int_0^1 (\log x) \frac{\log(1-x)}{x} dx.
\]
(5.7)
Comparing (5.7) with (5.3) completes the proof of (1.2).

5.5. Integration by Parts. Start with (5.7) and integrate by parts, obtaining
\[
2\zeta(3) = \int_0^1 \frac{\log^2 x}{1-x} dx = \int_0^1 \frac{\log^2(1-x)}{x} dx = \sum_{n,k>0} \int_0^1 \frac{x^{n+k-1}}{nk} dx = \sum_{n,k>0} \frac{1}{nk(n+k)}.
\]
Now see §2.

5.6. Triple Integrals I. This time, instead of (5.1) we use the elementary identity
\[
\frac{1}{k^2 n} = \int_0^1 y_1^{-1} \int_0^{y_1} y_2^{-n-1} \int_0^{y_2} y_3^{-1} dy_3 dy_2 dy_1, \quad k > n > 0.
\]
This yields
\[
\sum_{k>n>1} \frac{1}{k^2 n} = \sum_{k>n>0} \frac{1}{k^2 n} = \int_0^1 y_1^{-1} \int_0^{y_1} (1-y_2)^{-1} \int_0^{y_2} (1-y_3)^{-1} dy_3 dy_2 dy_1.
\]
(5.8)
Now make the change of variable $y_i = 1 - x_i$ for $i = 1, 2, 3$ to obtain
\[
\sum_{k>n>1} \frac{1}{k^2 n} = \int_0^1 (1-x_1)^{-1} \int_0^{x_1} x_2^{-1} \int_0^{x_2} x_3^{-1} dx_3 dx_2 dx_1
\]
\[= \int_0^1 x_3^{-1} \int_0^{x_3} x_2^{-1} \int_0^{x_2} (1-x_1)^{-1} dx_1 dx_2 dx_3.
\]
After expanding $(1-x_1)^{-1}$ into a geometric series and interchanging the order of summation and integration, one arrives at
\[
\sum_{k>n>1} \frac{1}{k^2 n} = \sum_{n=1}^{\infty} \int_0^1 x_3^{-1} \int_0^{x_3} x_2^{-1} \int_0^{x_2} x_1^{-1} dx_1 dx_2 dx_3 = \sum_{n=1}^{\infty} \frac{1}{n^3},
\]
as required.

More generally [9, 10, 11, 14, 15, 43],
\[
\zeta(s_1, \ldots, s_k) = \sum_{n_1 > \cdots > n_k > 0} \prod_{j=1}^k n_j^{-s_j} = \int \prod_{r=1}^k \frac{dt^{(j)}_{s_j}}{t^{(j)}_{s_j}} \frac{dt^{(j)}_{s_j}}{1-t^{(j)}_{s_j}},
\]
(5.9)
where the integral is over the simplex
\[ 1 > t_1^{(1)} > \cdots > t_k^{(1)} > \cdots > t_1^{(k)} > \cdots > t_k^{(k)} > 0, \]
and is abbreviated by
\[
\int_0^1 \prod_{j=1}^k a_j^{s_j - 1} b, \quad a = \frac{dt}{t}, \quad b = \frac{dt}{1 - t}. \tag{5.10}
\]

The change of variable \( t \mapsto 1 - t \) at each level of integration switches the differential forms \( a \) and \( b \), thus yielding the duality formula \([9] [43, p. 483]\) (conjectured in \([41]\))
\[
\zeta(s_1 + 2, \{1\}^{r_1}, \ldots, s_n + 2, \{1\}^{r_n}) = \zeta(r_n + 2, \{1\}^{s_n}, \ldots, r_1 + 2, \{1\}^{s_1}), \tag{5.11}
\]
which is valid for all nonnegative integers \( s_1, r_1, \ldots, s_n, r_n \). The case \( s_1 = 0, r_1 = 1 \) of (5.11) is (1.2). More generally, (1.4) can be restated as
\[
\int_0^1 (ab^2)^n = \int_0^1 (a^2b)^n
\]
and thus (1.4) is recovered by taking each \( s_j = 0 \) and each \( r_j = 1 \) in (5.11). For further generalizations and extensions of duality, see \([10, 19, 20]\).

For alternations, we require in addition the differential form \( c := -dt/(1 + t) \) with which we may form the generating function
\[
\sum_{n=1}^\infty z^{3n} \zeta(\{2, 1\}^n) = \sum_{n=0}^\infty \left\{ z^{6n+3} \int_0^1 (ac^2ab^2)^n ac^2 + z^{6n+6} \int_0^1 (ac^2ab^2)^{6n+6} \right\}.
\]
A lengthy calculation verifies that the only changes of variable that preserve the unit interval and send the non-commutative polynomial ring \( \mathbb{Q}(a, b) \) into \( \mathbb{Q}(a, b, c) \) are
\[
S(a, b) = S(a, b), \quad t \mapsto t, \tag{5.12}
\]
\[
S(a, b) = R(b, a), \quad t \mapsto 1 - t, \tag{5.13}
\]
\[
S(a, b) = S(2a, b - c), \quad t \mapsto t^2, \tag{5.14}
\]
\[
S(a, b) = S(a - c, b + c), \quad t \mapsto \frac{2t}{1 + t}, \tag{5.15}
\]
\[
S(a, b) = S(a - 2c, 2b + 2c), \quad t \mapsto \frac{4t}{(1 + t)^2}, \tag{5.16}
\]
and compositions thereof, such as \( t \mapsto 1 - 2t/(1 + t) = (1 - t)/(1 + t) \), etc. In (5.12)–(5.16), \( S(a, b) \) denotes a non-commutative word on the alphabet \( \{a, b\} \) and \( R(b, a) \) denotes the word formed by switching \( a \) and \( b \) and then reversing the order of the letters.

Now view \( a, b \) and \( c \) as indeterminates. In light of the polynomial \textit{identity}
\[
ab^2 - 8ac^2 = 2[ab^2 - 2a(b - c)^2] + 8[ab^2 - (a - c)(b + c)^2] + [(a - 2c)(2b + 2c)^2 - ab^2]
\]
in the non-commutative ring \( \mathbb{Z}\langle a, b, c \rangle \) and the transformations (5.14), (5.15) and (5.16) above, each bracketed term vanishes when we make the identifications \( a = dt/t \), \( b = dt/(1 - t) \), \( c = -dt/(1 + t) \) and perform the requisite iterated integrations. Thus,

\[
\zeta(2, 1) - 8\zeta(2, 1) = \int_0^1 ab^2 - 8 \int_0^1 ac^2 = 0,
\]

which in light of (1.2) proves (1.5). \( \square \)

5.7. **Triple Integrals II.** First, note that by expanding the integrands in geometric series and integrating term by term,

\[
\zeta(2, 1) = 8 \int_0^1 \frac{dx}{x} \int_0^x \frac{dy}{1 - y^2} \int_0^y \frac{dz}{1 - z^2}.
\]

Now make the change of variable

\[
\frac{x \, dx}{1 - x^2} = \frac{du}{1 + u}, \quad \frac{y \, dy}{1 - y^2} = \frac{dv}{1 + v}, \quad \frac{z \, dz}{1 - z^2} = \frac{dw}{1 + w}
\]

to obtain the equivalent integral

\[
\zeta(2, 1) = 8 \int_0^\infty \left( \frac{du}{2u} + \frac{du}{2(2 + u)} - \frac{du}{1 + u} \right) \int_u^\infty \frac{dv}{1 + v} \int_0^v \frac{dw}{1 + w}.
\]

The two inner integrals can be directly performed, leading to

\[
\zeta(2, 1) = 4 \int_0^\infty \frac{\log^2(u + 1)}{u(u + 1)(u + 2)} du.
\]

Finally, make the substitution \( u + 1 = 1/\sqrt{1 - x} \) to obtain

\[
\zeta(2, 1) = \frac{1}{2} \int_0^1 \frac{\log^2(1 - x)}{x} \, dx = \zeta(3),
\]

by (5.3).

5.8. **Complex Line Integrals I.** Here we apply the Mellin inversion formula [2, p. 243], [59, pp. 130–132 ]

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^z}{z} \, dz = \begin{cases} 
1, & y > 1 \\
0, & y < 1 \\
\frac{1}{2}, & y = 1
\end{cases}
\]
which is valid for fixed $c > 0$. It follows that if $c > 0$ and $s - 1 > c > 1 - t$ then the Perron-type formula

$$\zeta(s, t) + \frac{1}{2} \zeta(s + t) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} n^{-s} k^{-t} \int_{c - i \infty}^{c + i \infty} \left( \frac{n}{k} \right)^z dz$$

is valid. (Interchanging the order of summation and integration is permissible by absolute convergence.) Although we have not yet found a way to exploit (5.17) in proving identities such as (1.2), we note that by integrating around the rectangular contour with corners $(\pm c \pm iM)$ and then letting $M \to +\infty$, one can readily establish the stuffle formula in the form

$$\zeta(s, t) + \frac{1}{2} \zeta(s + t) + \zeta(t, s) + \frac{1}{2} \zeta(t + s) = \zeta(s) \zeta(t), \quad s, t > 1 + c.$$

The right hand side arises as the residue contribution of the integrand at $z = 0$. One can also use (5.17) to establish

$$\sum_{s=2}^{\infty} \left[ \zeta(s, 1) + \frac{1}{2} \zeta(s + 1) \right] x^{s-1} = \sum_{n>0} \sum_{m>0} x \frac{x}{mn(n-x)} + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{n>0} x \frac{x}{n(n-x)},$$

but this is easy to prove directly.

5.9. **Complex Line Integrals II.** We let $\lambda(s) := \sum_{n>0} \lambda_n n^{-s}$ represent a formal Dirichlet series, with real coefficients $\lambda_n$ and we set $s := \sigma + i \tau$ with $\sigma = \Re(s) > 0$, and consider the following integral:

$$\iota_\lambda(\sigma) := \int_0^\infty \left| \frac{\lambda(s)}{s} \right|^2 d\tau = \frac{1}{2} \int_{-\infty}^{\infty} \left| \frac{\lambda(s)}{s} \right|^2 d\tau, \quad (5.18)$$

as a function of $\lambda$. We begin with a useful variant of the Mellin inversion formula, namely

$$\int_0^\infty \frac{\cos(at)}{t^2 + u^2} dt = \frac{\pi}{2u} e^{-au}, \quad (5.19)$$

for $u, a > 0$, as follows by contour integration, from a computer algebra system, or otherwise. This leads to

**Theorem 2.** (Theorem 1 of [5]). For $\lambda(s) = \sum_{n>0} \lambda_n n^{-s}$ and $s = \sigma + i \tau$ with fixed $\sigma = \Re(s) > 0$, sufficiently large, we have

$$\iota_\lambda(\sigma) = \int_0^\infty \left| \frac{\lambda(s)}{s} \right|^2 d\tau = \frac{\pi}{2\sigma} \sum_{n>0} \frac{\lambda_n^2}{n^{2\sigma}} + \frac{\pi}{\sigma} \sum_{n>0} \frac{\lambda_n}{n^{2\sigma}} + \frac{\pi}{\sigma} \sum_{m=1}^{n-1} \lambda_m \frac{\sum_{m=1}^{n-1} \lambda_m}{n^{2\sigma}} = \frac{\pi}{\sigma} \sum_{n>0} \frac{\lambda_n \Lambda_n}{n^{2\sigma}}, \quad (5.20)$$
where \( \Lambda_n := \sum_{m=1}^{n-1} \lambda_m + \lambda_n/2 \). More generally, given two L-series \( \alpha(s) = \sum_{n>0} \alpha_n n^{-s} \) and \( \beta(s) = \sum_{n>0} \beta_n n^{-s} \) we have

\[
\int_0^\infty \frac{\alpha(s) \overline{\beta}(s)}{\sigma^2 + \tau^2} d\tau = \frac{\pi}{2\sigma} \sum_{n>0} \frac{\alpha_n B_n}{n^{2\sigma}} + \frac{\pi}{2\sigma} \sum_{n>0} \frac{\beta_n A_n}{n^{2\sigma}} + \frac{\pi}{2\sigma} \sum_{n>0} \frac{\alpha_n \beta_n}{n^{2\sigma}}
\]

(5.21)

where \( A_n := \sum_{m=1}^{n-1} \alpha_m \) and \( B_n := \sum_{m=1}^{n-1} \beta_m \).

Note that the righthand side of (5.20) is always a generalized Euler sum.

For the Riemann zeta function, and for \( \sigma > 1 \), Theorem 2 applies and yields

\[
\frac{\sigma}{\pi} \iota_\zeta(\sigma) = \zeta(2\sigma - 1) - \frac{1}{2} \zeta(2\sigma),
\]

as \( \lambda_n = 1 \) and \( \Lambda_n = n - 1/2 \). By contrast it is known that on the critical line

\[
\frac{1/2}{\pi} \iota_\zeta \left( \frac{1}{2} \right) = \log(\sqrt{2\pi}) - \frac{1}{2} \gamma.
\]

There are similar formulae for \( s \mapsto \zeta(s - k) \) with \( k \) integral. For instance, applying the result in (5.20) with \( \zeta_1 := t \mapsto \zeta(t + 1) \) yields

\[
\frac{1}{\pi} \int_0^\infty \frac{|\zeta(3/2 + i\tau)|^2}{1/4 + \tau^2} d\tau = \frac{1}{\pi} \iota_{\zeta_1} \left( \frac{1}{2} \right) = 2 \zeta(2,1) + \zeta(3) = 3 \zeta(3),
\]

on using (1.2). For the alternating zeta function, \( \alpha := s \mapsto (1 - 2^{1-s})\zeta(s) \), the same approach via (5.21) produces

\[
\frac{1}{\pi} \int_0^\infty \frac{\alpha(3/2 + i\tau) \overline{\alpha}(3/2 + i\tau)}{1/4 + \tau^2} d\tau = 2 \zeta(2,1) + \zeta(3) = 3 \zeta(2) \log(2) - \frac{9}{4} \zeta(3),
\]

and

\[
\frac{1}{\pi} \int_0^\infty \frac{\alpha(3/2 + i\tau) \overline{\zeta}(3/2 + i\tau)}{1/4 + \tau^2} d\tau = \zeta(2,1) + \zeta(2,1) + \alpha(3) = \frac{9}{8} \zeta(2) \log(2) - \frac{3}{4} \zeta(3),
\]

since as we have seen repeatedly \( \zeta(2,1) = \zeta(3)/8 \); while \( \zeta(2,1) = \zeta(3) - 3/2 \zeta(2) \log(2) \) and \( \zeta(2,1) = 3/2 \zeta(2) \log(2) - 13/8 \zeta(3) \), (e.g., [12]).

As in the previous subsection we have not been able to directly obtain (1.5) or even (1.2), but we have connected them to quite difficult line integrals.
5.10. **Contours Integrals and Residues.** Following [57], let \( C_n \) (\( n \in \mathbb{Z}^+ \)) be the square contour with vertices \((\pm 1 \pm i)(n + 1/2)\). Using the asymptotic expansion

\[
\psi(z) \sim \log z - \frac{1}{2z} - \sum_{r=1}^{\infty} \frac{B_{2r}}{2rz^{2r}}, \quad |\arg z| < \pi
\]

in terms of the Bernoulli numbers

\[
\frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} t^{2r}, \quad |t| < 2\pi
\]

and the identity

\[
\psi(z) = \psi(-z) - \frac{1}{z} - \pi \cot \pi z,
\]

we can show that for each integer \( k \geq 2 \),

\[
\lim_{n \to \infty} \int_{C_n} z^{-k} \psi^2(-z) \, dz = 0.
\]

Then by the residue theorem, we obtain

**Theorem 3** (Theorem 3 of [57]). For every integer \( k \geq 2 \),

\[
2 \sum_{n=1}^{\infty} n^{-k} \psi(n) = k \zeta(k + 1) - 2\gamma \zeta(k) - \sum_{j=1}^{k-1} \zeta(j) \zeta(k - j + 1),
\]

where \( \gamma = 0.577215664\ldots \) is Euler’s constant.

In light of the identity

\[
\psi(n) + \gamma = H_{n-1} = \sum_{k=1}^{n-1} \frac{1}{k}, \quad n \in \mathbb{Z}^+,
\]

Theorem 3 is equivalent to (1.3). The case \( k = 2 \) thus gives (1.2).

Flajolet and Salvy [37] developed the residue approach more systematically, and applied it to a number of other Euler sum identities in addition to (1.3).

6. **A Stirling Number Generating Function**

Following [28], we begin with the integral representation (5.5) of §5.2. In light of the expansion

\[
\frac{(-1)^m}{m!} \log^m(1 - x) = \sum_{n=0}^{\infty} u(n, m) \frac{x^n}{n!}, \quad 0 \leq m \in \mathbb{Z},
\]

where \( u(n, m) \) is the Stirling number of the second kind.
in terms of the unsigned Stirling numbers of the first kind (also referred to as the Stirling cycle numbers in [39]), we have

$$
\zeta(m + 1) = \int_0^1 \left\{ \sum_{n=1}^{\infty} u(n, m) \frac{x^n}{n!} \right\} \frac{dx}{x} = \sum_{n=1}^{\infty} \frac{u(n, m)}{n!}, \quad 1 \leq m \in \mathbb{Z}.
$$

Telescoping the known recurrence

$$
u(n, m) = u(n - 1, m - 1) + (n - 1)u(n - 1, m), \quad 1 \leq m \leq n,
$$

yields

$$
u(n, m) = (n - 1)! \left\{ \delta_{m, 1} + \sum_{j=1}^{n-1} \frac{u(j, m - 1)}{j!} \right\}. \quad (6.2)
$$

Iterating this gives the representation

$$
\zeta(m + 1) = \zeta(2, \{1\}^{m-1}), \quad 1 \leq m \in \mathbb{Z},
$$

the $m = 2$ case of which is [1.2]. See also $n = 0$ in (9.3) below. \qed

For the alternating case, we begin by writing the recurrence (6.1) in the form

$$
u(n + 1, k) + (j - n)u(n, k) = u(n, k - 1) + j u(n, k).
$$

Following [28], multiply both sides by $(-1)^{n+k+1} j^{k-m-1}/(j-n)_n$, where $1 \leq n \leq j - 1$ and $k, m \in \mathbb{Z}^+$, yielding

$$
(-1)^k \left\{ \frac{(-1)^{n+1} u(n + 1, k)}{(j-n)_n} - \frac{(-1)^n u(n, k)}{(j-n+1)_{n-1}} \right\} j^{k-m-1} = \frac{(-1)^n}{(j-n)_n} \left\{ (-1)^{k-1} u(n, k - 1) j^{k-m-1} - (-1)^k u(n, k) j^{k-m} \right\}.
$$

Now sum on $1 \leq k \leq m$ and $1 \leq n \leq j - 1$, obtaining

$$
\sum_{k=1}^{m} \frac{(-1)^{k+j} u(j, k)}{j! j^{m-k}} - \frac{1}{j^m} = \frac{(-1)^{m+1}}{(j-1)!} \sum_{n=m}^{j-1} (-1)^n (j-n-1)! u(n, m).
$$

Finally, sum on $j \in \mathbb{Z}^+$ to obtain

$$
\zeta(m) = \sum_{k=1}^{m} \sum_{j=k}^{\infty} \frac{(-1)^{k+j} u(j, k)}{j! j^{m-k}} + \sum_{n=m}^{\infty} (-1)^{n+m} u(n, m) \sum_{j=n+1}^{\infty} \frac{(j-1-n)!}{(j-1)!}.
$$

Noting that

$$
\sum_{j=n+1}^{\infty} \frac{(j-1-n)!}{(j-1)!} = \sum_{k=0}^{\infty} \frac{k!}{(k+n)!} = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{k!}{(k+n)!} 2F_1(1, 1; n+1; 1) = \frac{1}{(n-1)! (n-1)},
$$
we find that
\[ \zeta(m) = \sum_{k=1}^{m} \sum_{j=k}^{\infty} \frac{(-1)^{j+k} u(j, k)}{j! j^{m-k}} + \sum_{n=m}^{\infty} \frac{(-1)^{n+m} u(n, m)}{(n-1)! (n-1)}. \]

Now employ the recurrence (6.1) again to get
\[ \zeta(m) = \sum_{k=1}^{m-2} \sum_{j=k}^{\infty} \frac{(-1)^{j+k} u(j, k)}{j! j^{m-k}} + \sum_{j=m-1}^{\infty} \frac{(-1)^{j+m-1} u(j, m-1)}{j! j} + \sum_{j=m}^{\infty} \frac{(-1)^{j+m} u(j, m)}{j! j} \\
+ \sum_{n=m}^{\infty} \frac{(-1)^{n+m} u(n-1, m)}{(n-1)! (n-1)} + \sum_{n=m}^{\infty} \frac{(-1)^{n+m} u(n-1, m-1)}{(n-1)! (n-1)} \\
= \sum_{k=1}^{m-2} \sum_{j=k}^{\infty} \frac{(-1)^{j+k} u(j, k)}{j! j^{m-k}} + 2 \sum_{j=m-1}^{\infty} \frac{(-1)^{j+m-1} u(j, m-1)}{j! j}. \tag{6.3} \]

Using (6.2) again, we find that the case \( m = 3 \) of (6.3) gives
\[ \zeta(3) = \sum_{j=1}^{\infty} \frac{(-1)^{j} u(j, 1)}{j! j^2} + 2 \sum_{j=2}^{\infty} \frac{(-1)^{j} u(j, 2)}{j! j} \\
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^3} + 2 \sum_{j=2}^{\infty} \frac{(-1)^{j+1} (j-1)! \sum_{k=1}^{j-1} u(k, 1)}{k!} \\
= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^3} + 2 \sum_{j=2}^{\infty} \frac{(-1)^{j+1} j^{-1} \sum_{k=1}^{j-1} 1}{j^2} \\
= 2\zeta(2, 1) - \zeta(3), \]
which easily rearranges to give (2.3), shown in §2 to be trivially equivalent to (1.5). \( \square \)

7. Polylogarithm Identities

7.1. Dilogarithm and Trilogarithm. Consider the power series
\[ J(x) := \zeta_x(2, 1) = \sum_{n>k>0} \frac{x^n}{n^2 k}, \quad 0 \leq x \leq 1. \]

In light of (11.3), we have
\[ J(x) = \int_0^x \frac{dt}{t} \int_0^t \frac{du}{1-u} \int_0^u \frac{dv}{1-v} = \int_0^x \frac{\log^2(1-t)}{2t} dt. \]

The computer algebra package MAPLE readily evaluates
\[ \int_0^x \frac{\log^2(1-t)}{2t} dt = \zeta(3) + \frac{1}{2} \log^2(1-x) \log(x) + \log(1-x) \text{Li}_2(1-x) - \text{Li}_3(1-x) \tag{7.1} \]
where
\[ \text{Li}_s(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^s} \]
is the classical polylogarithm [47, 48]. (One can also readily verify the identity (7.1) by differentiating both sides by hand, and then checking (7.1) trivially holds as \( x \to 0^+ \). See also [4, p. 251, Entry 9].) Thus,
\[ J(x) = \zeta(3) + \frac{1}{2} \log^2(1-x) \log(x) + \log(1-x) \text{Li}_2(1-x) - \text{Li}_3(1-x). \]
Letting \( x \to 1^- \) gives (1.2) again. \( \Box \)

In [4, p. 251, Entry 9], we also find that
\[ J(-z) + J(-1/z) = -\frac{1}{6} \log^3 z - \text{Li}_2(-z) \log z + \text{Li}_3(-z) + \zeta(3) \]
and
\[ J(1-z) = \frac{1}{2} \log^2 z \log(z-1) - \frac{1}{8} \log^3 z - \text{Li}_2(1/z) \log z - \text{Li}_3(1/z) + \zeta(3). \]
Putting \( z = 1 \) in (7.2) and employing the well-known dilogarithm evaluation [47, p. 4]
\[ \text{Li}_2(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12} \]
gives (1.5). Putting \( z = 2 \) in (7.3) and employing the dilogarithm evaluation [47, p. 6]
\[ \text{Li}_2 \left( \frac{1}{2} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n} = \frac{\pi^2}{12} - \frac{1}{2} \log^2 2 \]
and the trilogarithm evaluation [47, p. 155]
\[ \text{Li}_3 \left( \frac{1}{2} \right) = \sum_{n=1}^{\infty} \frac{1}{n^3 2^n} = \frac{7}{8} \zeta(3) - \frac{\pi^2}{12} \log 2 + \frac{1}{6} \log^3 2 \]
gives (1.5) again. \( \Box \)

Finally, as in [10] Lemma 10.1], differentiation shows that
\[ J(-x) = -J(x) + \frac{1}{4} J(x^2) + J \left( \frac{2x}{x+1} \right) - \frac{1}{8} J \left( \frac{4x}{(x+1)^2} \right). \]
Putting [10] Theorem 10.3] \( x = 1 \) gives \( 8J(-1) = J(1) \) immediately, i.e. (1.3). \( \Box \)

In [10], it is noted that once the component functions in (7.4) are known, the coefficients can be deduced by computing each term to high precision with a common transcendental value of \( x \) and then employing a linear relations finding algorithm. We note here a somewhat more satisfactory method for arriving at (7.4).

First, as in [5,6] one must determine the fundamental transformations (5.12)–(5.16). While this is not especially difficult, as the calculations are somewhat lengthy, we do not
include them here. By performing these transformations on the function $J(x)$, one finds that

$$J(x) = \int_0^x ab^2, \quad J \left( \frac{2x}{1 + x} \right) = \int_0^x (a - c)(b + c)^2,$$

$$J(-x) = \int_0^x ac^2, \quad J \left( \frac{4x}{(1 + x)^2} \right) = \int_0^x (a - 2c)4(b + c)^2.$$

It now stands to reason that we should seek rational numbers $r_1, r_2, r_3$ and $r_4$ such that

$$ac^2 = r_1 ab^2 + 2r_2 a(b - c)^2 + r_3(a - c)(b + c)^2 + r_4(a - 2c)4(b + c)^2$$

is an identity in the non-commutative polynomial ring $\mathbb{Q} \langle a, b, c \rangle$. The problem of finding such rational numbers reduces to solving a finite set of linear equations. For example, comparing coefficients of the monomial $ab^2$ tells us that $r_1 = -1$, $r_2 = 1/4$, $r_3 = 1$ and $r_4 = -1/8$, thus proving (7.4) as expected.

7.2. Convolution of Polylogarithms. Motivated by [26, 27], for real $0 < x < 1$ and integers $s$ and $t$, consider

$$T_{s,t}(x) := \sum_{m,n=1 \atop m \neq n}^{\infty} \frac{x^{n+m}}{n^s m^t (m-n)} = \sum_{m,n=1 \atop m \neq n}^{\infty} \frac{x^{n+m} (m-n+n)}{n^s m^{t+1} (m-n)}$$

$$= \sum_{m,n=1 \atop m \neq n}^{\infty} \frac{x^{n+m}}{n^s m^{t+1}} + \sum_{m,n=1 \atop m \neq n}^{\infty} \frac{x^{n+m}}{n^{s-1} m^{t+1} (m-n)}$$

$$= \sum_{n=1}^{\infty} x^n \sum_{m=1}^{\infty} \left( \frac{x^m}{m^{t+1}} - \frac{x^n}{n^{t+1}} \right) + T_{s-1,t+1}(x)$$

$$= \text{Li}_s(x)\text{Li}_{t+1}(x) - \text{Li}_{s+t+1}(x^2) + T_{s-1,t+1}(x).$$

Telescoping this gives

$$T_{s,t}(x) = T_{0,s+t}(x) - s \text{Li}_{s+t+1}(x^2) + \sum_{j=1}^{s} \text{Li}_j(x)\text{Li}_{s+t+1-j}(x), \quad 0 \leq s \in \mathbb{Z}.$$

With $t = 0$, this becomes

$$T_{s,0}(x) = T_{0,s}(x) - s \text{Li}_{s+1}(x^2) + \sum_{j=1}^{s} \text{Li}_j(x)\text{Li}_{s+1-j}(x), \quad 0 \leq s \in \mathbb{Z}.$$
But for any integers \(s\) and \(t\), there holds
\[
T_{s,t}(x) = \sum_{m,n=1}^{\infty} \frac{x^{n+m}}{m^s n^t (m-n)} = - \sum_{m,n=1}^{\infty} \frac{x^{n+m}}{m^s n^t (n-m)} = -T_{s,t}(x).
\]

Therefore,
\[
T_{s,0}(x) = \frac{1}{2} \sum_{j=1}^{s} \text{Li}_j(x) \text{Li}_{s+1-j}(x) - \frac{s}{2} \text{Li}_{s+1}(x^2), \quad 0 \leq s \in \mathbb{Z}. \tag{7.5}
\]

On the other hand,
\[
T_{s,0}(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{m\neq n} \frac{x^m}{m-n} = \sum_{n=1}^{\infty} \frac{x^{2n}}{n^s} \sum_{m=n+1}^{\infty} \frac{x^{m-n}}{m-n} - \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{m=1}^{n-1} \frac{x^m}{n-m}
\]
\[
= \text{Li}_s(x^2) \text{Li}_1(x) - \sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{j=1}^{n-1} \frac{x^{n-j}}{j}.
\]

Comparing this with (7.5) gives
\[
\sum_{n=1}^{\infty} \frac{x^n}{n^s} \sum_{j=1}^{n-1} \frac{x^{n-j}}{j} = \frac{s}{2} \text{Li}_{s+1}(x^2) - \left[\text{Li}_s(x) - \text{Li}_s(x^2)\right] \text{Li}_1(x) - \frac{1}{2} \sum_{j=2}^{s-1} \text{Li}_j(x) \text{Li}_{s+1-j}(x), \tag{7.6}
\]

where in (7.6) and what follows, we now require \(2 \leq s \in \mathbb{Z}\) because the terms \(j = 1\) and \(j = s\) in the sum (7.5) were separated, and assumed to be distinct.

Next, note that if \(n\) is a positive integer and \(0 < x < 1\), then
\[
1 - x^n = (1 - x) \sum_{j=0}^{n-1} x^j < (1 - x)n.
\]

Thus, if \(2 \leq s \in \mathbb{Z}\) and \(0 < x < 1\), then
\[
0 < \left[\text{Li}_s(x) - \text{Li}_s(x^2)\right] \text{Li}_1(x) = \text{Li}_1(x) \sum_{n=1}^{\infty} \frac{x^n(1-x^n)}{n^s} < (1 - x) \text{Li}_1(x) \sum_{n=1}^{\infty} \frac{x^n}{n^{s-1}}
\]
\[
< (1 - x) \log^2(1 - x).
\]

Since the latter expression tends to zero in the limit as \(x \to 1^-\), taking the limit in (7.6) gives
\[
\zeta(s,1) = \frac{1}{2} s \zeta(s+1) - \frac{1}{2} \sum_{j=1}^{s-2} \zeta(j+1) \zeta(s-j), \quad 2 \leq s \in \mathbb{Z},
\]
which is \( (1.3) \).

\[ \square \]

8. Fourier Series

The Fourier expansions
\[
\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} = \frac{\pi - t}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(nt)}{n} = -\log|2\sin(t/2)|
\]
are both valid in the open interval \( 0 < t < 2\pi \). Multiplying these together, simplifying, and doing a partial fraction decomposition gives
\[
\sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \left( \frac{1}{n} - \frac{1}{k} \right) = \frac{1}{2} \sum_{n=k=0}^{\infty} \frac{\sin(nt)}{k(n-k)} = \frac{1}{2} \sum_{m,n=1}^{\infty} \frac{\sin(mt) \cos(nt)}{mn} \sum_{n-k>0}^{\infty} \sin(nt)
\]
again for \( 0 < t < 2\pi \). Integrating (8.1) term by term yields
\[
\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^2} = \frac{\zeta(2,1)}{2} + \frac{1}{2} \int_0^\theta (\pi - t) \log|2\sin(t/2)| \, dt, \tag{8.2}
\]
valid for \( 0 \leq \theta \leq 2\pi \). Likewise for \( 0 \leq \theta \leq 2\pi \),
\[
\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n^3} = \frac{\zeta(3)}{128} + \frac{1}{2} \int_0^\theta (\theta - t) \log|2\sin(t/2)| \, dt. \tag{8.3}
\]
Setting \( \theta = \pi \) in (8.2) and (8.3) produces
\[
\zeta(2,1) - \zeta(2,1) = -\frac{1}{2} \int_0^\pi (\pi - t) \log|2\sin(t/2)| \, dt = \frac{\zeta(3) - \zeta(3)}{2}.
\]
In light of (1.2), this implies
\[
\zeta(2,1) = \frac{\zeta(3) + \zeta(3)}{2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n^3} = \sum_{m=1}^{\infty} \frac{1}{(2m)^3} = \frac{1}{8} \zeta(3),
\]
which is (1.5).

Applying Parseval’s equation to (8.1) gives (via [6, 8, 37]) the integral evaluation
\[
\frac{1}{4\pi} \int_0^{2\pi} (\pi - t)^2 \log^2(2\sin(t/2)) \, dt = \sum_{n=1}^{\infty} \frac{H_n^2}{(n+1)^2} = \frac{11}{4} \zeta(4).
\]
A reason for valuing such integral representations is that they are frequently easier to use numerically.

9. Further Generating Functions

9.1. Hypergeometric Functions. Note that in the notation of (1.1), \( \zeta(2, 1) \) is the coefficient of \( x y^2 \) in

\[
G(x, y) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+1} y^{n+1} \zeta(m + 2, \{1\}^n) = y \sum_{m=0}^{\infty} x^{m+1} \sum_{k=1}^{\infty} \frac{1}{k^{m+2}} \prod_{j=1}^{k-1} \left(1 + \frac{y}{j}\right). \tag{9.1}
\]

Now recall the notation \( (y)_k := y(y + 1) \cdots (y + k - 1) \) for the rising factorial with \( k \) factors. Thus,

\[
\frac{y}{k} \prod_{j=1}^{k-1} \left(1 + \frac{y}{j}\right) = \frac{(y)_k}{k!}.
\]

Substituting this into (9.1), interchanging order of summation, and summing the resulting geometric series yields the hypergeometric series

\[
G(x, y) = \sum_{k=1}^{\infty} \frac{(y)_k}{k!} \left(\frac{x}{k - x}\right) = -\sum_{k=1}^{\infty} \frac{(y)_k (-x)_k}{k! (1 - x)_k} = 1 - {}_2F_1 \left( \begin{array}{c} y, -x \\ 1 - x \end{array} \right) \left| 1 \right).
\]

But, Gauss’s summation theorem for the hypergeometric function [11, p. 557] [3, p. 2] and the power series expansion for the logarithmic derivative of the gamma function [11, p. 259] imply that

\[
{}_2F_1 \left( \begin{array}{c} y, -x \\ 1 - x \end{array} \right) \left| 1 \right) = \frac{\Gamma(1 - x) \Gamma(1 - y)}{\Gamma(1 - x - y)} = \exp \left\{ \sum_{k=2}^{\infty} \left( x^k + y^k - (x + y)^k \right) \frac{\zeta(k)}{k} \right\}.
\]

Thus, we have derived the generating function equality [9] (see [19] for a \( q \)-analog)

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^{m+1} y^{n+1} \zeta(m + 2, \{1\}^n) = 1 - \exp \left\{ \sum_{k=2}^{\infty} \left( x^k + y^k - (x + y)^k \right) \frac{\zeta(k)}{k} \right\}. \tag{9.2}
\]

Extracting coefficients of \( x y^2 \) from both sides of (9.2) yields (1.2).

The generalization (1.3) can be similarly derived: extract the coefficient of \( x^{m-1} y^2 \) from both sides of (9.2). In fact, it is easy to see that (9.2) provides a formula for \( \zeta(m + 2, \{1\}^n) \) for all nonnegative integers \( m \) and \( n \) in terms of sums of products of values of the Riemann zeta function at the positive integers. In particular, Markett’s formula [49] (cf. also [8]) for \( \zeta(m, 1, 1) \) for positive integers \( m > 1 \) is most easily obtained in this way. Noting symmetry between \( x \) and \( y \) in (9.2) gives Drinfeld’s duality formula [33]

\[
\zeta(m + 2, \{1\}^n) = \zeta(n + 2, \{1\}^m) \tag{9.3}
\]
for non-negative integers $m$ and $n$, a special case of the more general duality formula (5.11). Note that (1.2) is just the case $m = n = 0$.

Similarly [30 2.1b] equating coefficients of $xy^2$ in Kummer’s summation theorem [44, p. 53] [3, p. 9]

$$2F_1\left(\begin{array}{c} x, y \\ 1 + x - y \end{array} \bigg| -1 \right) = \frac{\Gamma(1 + x/2)\Gamma(1 + x - y)}{\Gamma(1 + x)\Gamma(1 + x/2 - y)}$$

yields (1.5).

9.2. A Generating Function for Sums. The identity (1.2) can also be recovered by setting $x = 0$ in the following result:

**Theorem 4** (Theorem 1 of [7]). If $x$ is any complex number not equal to a positive integer, then

$$\sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m-x} = \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}.$$  

**Proof.** Fix $x \in \mathbb{C} \setminus \mathbb{Z}^+$. Let $S$ denote the left hand side. By partial fractions,

$$S = \sum_{n=1}^{\infty} \sum_{m=1}^{n-1} \left( \frac{1}{n(n-m)(m-x)} - \frac{1}{n(n-m)(n-x)} \right)$$

$$= \sum_{m=1}^{\infty} \frac{1}{m-x} \sum_{n=m+1}^{\infty} \frac{1}{n(n-m)} - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{n-m}$$

$$= \sum_{m=1}^{\infty} \frac{1}{m(m-x)} \sum_{n=m+1}^{\infty} \left( \frac{1}{n-m} - \frac{1}{n} \right) - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m}.$$  

Now for fixed $m \in \mathbb{Z}^+$,

$$\sum_{n=m+1}^{\infty} \left( \frac{1}{n-m} - \frac{1}{n} \right) = \lim_{N \to \infty} \sum_{n=m+1}^{N} \left( \frac{1}{n-m} - \frac{1}{n} \right) = \sum_{n=1}^{m} \frac{1}{n} - \lim_{N \to \infty} \sum_{n=1}^{N} \frac{m}{N-n+1}$$

$$= \sum_{n=1}^{m} \frac{1}{n},$$

since $m$ is fixed. Therefore, we have

$$S = \sum_{m=1}^{\infty} \frac{1}{m(m-x)} \sum_{n=1}^{m} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \sum_{m=1}^{n-1} \frac{1}{m} = \sum_{n=1}^{\infty} \frac{1}{n(n-x)} \left( \sum_{m=1}^{n} \frac{1}{m} - \sum_{m=1}^{n-1} \frac{1}{m} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2(n-x)}.$$  

$\square$
Theorem 4 is in fact equivalent to the sum formula \[ \sum_{a_i+s} \zeta(a_1+2,a_2+1,\ldots,a_r+1) = \zeta(r+s+1), \] valid for all integers \( s \geq 0, r \geq 1, \) and which generalizes Theorem 1 (4.1) to arbitrary depth. The identity (1.2) is simply the case \( r = 2, s = 0. \) A \( q \)-analog of the sum formula (9.4) is derived as a special case of more general results in [19]. See also [21].

9.3. **An Alternating Generating Function.** An alternating counterpart to Theorem 4 is given below.

**Theorem 5.** (Theorem 3 of [7].) For all non-integer \( x \)

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left( H_n + \sum_{n=1}^{\infty} \frac{x^2}{n(n^2 - x^2)} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left( \psi(n) - \psi(x) - \frac{\pi}{2} \cot(\pi x) - \frac{1}{2x} \right)
\]

\[
= \sum_{o>0, \text{odd}} \frac{1}{o(o^2 - x^2)} + \sum_{n=1}^{\infty} \frac{(-1)^n n}{(n^2 - x^2)^2}
\]

\[
= \sum_{e>0, \text{even}} \frac{e}{(x^2 - e^2)^2} - x^2 \sum_{o>0, \text{odd}} \frac{1}{o(x^2 - o^2)^2}.
\]

Setting \( x = 0 \) reproduces (1.5) in the form \( \zeta(2,1) = \sum_{n>0}^{\infty} (2n)^{-3}. \) We record that

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} = \frac{1}{2x^2} - \frac{\pi}{2x} \sin(\pi x),
\]

while

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left( \psi(n) - \psi(x) - \frac{\pi}{2} \cot(\pi x) - \frac{1}{2x} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 - x^2} \left( H_n + \sum_{n=1}^{\infty} \frac{x^2}{n(n^2 - x^2)} \right)
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{(2n-1)((2n-1)^2 - x^2)} + \sum_{n=1}^{\infty} \frac{n(-1)^n}{(n^2 - x^2)^2}.
\]

9.4. **The Digamma Function.** Define an auxiliary function \( \Lambda \) by

\[
x\Lambda(x) := \frac{1}{2} \psi'(1-x) - \frac{1}{2} (\psi(1-x) + \gamma)^2 - \frac{1}{2} \zeta(2).
\]

We note, but do not use, that

\[
x\Lambda(x) = \frac{1}{2} \int_0^{\infty} \frac{t(1-x)(1+t(1-x))}{1-e^{-t}} dt - \frac{1}{2} \left( \int_0^{\infty} \frac{e^{-t} - e^{-t(1-x)}}{1-e^{-t}} dt \right)^2 - \zeta(2).
\]
It is easy to verify that
\[
\psi(1 - x) + \gamma = \sum_{n=1}^{\infty} \frac{x}{n(x - n)},
\]
\[
\psi'(1 - x) - \zeta(2) = \sum_{n=1}^{\infty} \left( \frac{1}{(x - n)^2} - \frac{1}{n^2} \right) = \sum_{n=1}^{\infty} \frac{2nx - x^2}{n^2(n - x)^2},
\]
(9.5)
and
\[
\sum_{n=0}^{\infty} \zeta(n + 2, 1)x^n = \sum_{n=1}^{\infty} \frac{1}{n(n - x)} \sum_{m=1}^{n-1} \frac{1}{m}.
\]
Hence,
\[
\Lambda(x) = \sum_{n=1}^{\infty} \frac{1}{n^2(n - x)} - x \sum_{n=1}^{\infty} \frac{1}{n(n - x)} \sum_{m=1}^{n-1} \frac{1}{m(m - x)}.
\]
Now,
\[
\sum_{n=1}^{\infty} \frac{1}{n^2(n - x)} - x \sum_{n=1}^{\infty} \frac{1}{n(n - x)} \sum_{m=1}^{n-1} \frac{1}{m(m - x)} = \sum_{n=1}^{\infty} \frac{1}{n(n - x)} \sum_{m=1}^{n-1} \frac{1}{m}
\]
is directly equivalent to Theorem 4 of §9.2—see [7, Section 3]—and we have proven
\[
\Lambda(x) = \sum_{n=0}^{\infty} \zeta(n + 2, 1)x^n,
\]
so that comparing coefficients yields yet another proof of Euler’s reduction (1.3). In particular, setting \(x = 0\) again produces (1.2).

9.5. The Beta Function. Recall that the beta function is defined for positive real \(x\) and \(y\) by
\[
B(x, y) := \int_{0}^{1} t^{x-1}(1 - t)^{y-1} \, dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.
\]
We begin with the following easily obtained generating function:
\[
\sum_{n=1}^{\infty} t^n H_n = -\frac{\log(1 - t)}{1 - t}.
\]
For \( m \geq 2 \) the Laplace integral (5.4) now gives
\[
\zeta(m, 1) = \frac{(-1)^m}{(m-1)!} \int_0^1 \log^{m-1}(t) \frac{\log(1-t)}{1-t} \, dt
= \frac{(-1)^m}{2(m-1)!} \int_0^1 (m-1) \log^{m-2}(t) \log^2(1-t) \, \frac{dt}{t}
= \frac{(-1)^m}{2(m-2)!} b_1^{(m-2)}(0),
\]
where
\[
b_1(x) := \left. \frac{\partial^2}{\partial y^2} B(x,y) \right|_{y=1} = 2\Lambda(-x)
\]
(cf. (9.4)). Since
\[
\frac{\partial^2}{\partial y^2} B(x,y) = B(x,y) \left[ (\psi(y) - \psi(x+y))^2 + \psi'(y) - \psi'(x+y) \right],
\]
we derive
\[
b_1(x) = \frac{(\psi(1) - \psi(x+1))^2 + \psi'(1) - \psi'(x+1)}{x}.
\]
Now observe that from (9.6),
\[
\zeta(2, 1) = \frac{1}{2} b_1(0) = \lim_{x \downarrow 0} \frac{(-\gamma - \psi(x+1))^2}{2x} - \lim_{x \downarrow 0} \frac{\psi'(x+1) - \psi'(1)}{2x} = -\frac{1}{2} \psi''(1)
= \zeta(3).
\]
Continuing, from the following two identities, cognate to (9.4),
\[
(-\gamma - \psi(x+1))^2 = \left( \sum_{m=1}^{\infty} (-1)^m \zeta(m+1) x^m \right)^2
= \sum_{m=1}^{\infty} (-1)^m \sum_{k=1}^{m-1} \zeta(k+1) \zeta(m-k+1) x^m,
\]
\[
\zeta(2) - \psi'(x+1) = \sum_{m=1}^{\infty} (-1)^{m+1} (m+1) \zeta(m+2) x^m,
\]
we get
\[
2 \sum_{m=2}^{\infty} (-1)^m \zeta(m, 1) x^{m-2} = \sum_{m=2}^{\infty} b_1^{(m-2)}(0) x^{m-2} = b_1(x)
= \sum_{m=1}^{\infty} (-1)^{m-1} \left( (m+1) \zeta(m+2) - \sum_{k=1}^{m-1} \zeta(k+1) \zeta(m-k+1) \right) x^{m-1},
\]
from which Euler’s reduction (1.3) follows—indeed this is close to Euler’s original path.

Observe that (9.6) is especially suited to symbolic computation. We also note the pleasing identity

\[ \psi'(x) = \frac{\Gamma''(x)}{\Gamma(x)} - \psi^2(x). \]  

(9.7)

In some informal sense (9.7) generates (1.3), but we have been unable to make this sense precise.

10. A Decomposition Formula of Euler

For positive integers \( s \) and \( t \) and distinct non-zero real numbers \( \alpha \) and \( x \), the partial fraction expansion

\[
\frac{1}{x^s(x-\alpha)^t} = (-1)^t \sum_{r=0}^{s-1} \binom{t+r-1}{t-1} \frac{1}{x^{s-r} \alpha^{t+r} + \sum_{r=0}^{t-1} \binom{s+r-1}{s-1} \frac{(-1)^r}{\alpha^{s+r}(x-\alpha)^{t-r}}
\]

(10.1)

implies [52, p. 48] [49] Euler’s decomposition formula

\[
\zeta(s, t) = (-1)^t \sum_{r=0}^{s-2} \left( \begin{array}{c} t+r-1 \\ t-1 \end{array} \right) \zeta(s-r, t+r) + \sum_{r=0}^{t-2} (-1)^r \left( \begin{array}{c} s+r-1 \\ s-1 \end{array} \right) \zeta(t-r) \zeta(s+r)
\]

\[ - (-1)^t \left( \begin{array}{c} s+t-2 \\ s-1 \end{array} \right) \left\{ \zeta(s+t) + \zeta(s+t-1, 1) \right\}. \]  

(10.2)

The depth-2 sum formula (10.1) is obtained by setting \( t = 1 \) in (10.2). If we also set \( s = 2 \), the identity (1.2) results. To derive (10.2) from (10.1) we follow [49], separating the last term of each sum on the right hand side of (10.1), obtaining

\[
\frac{1}{x^s(x-\alpha)^t} = (-1)^t \sum_{r=0}^{s-2} \left( \begin{array}{c} t+r-1 \\ t-1 \end{array} \right) \frac{1}{x^{s-r} \alpha^{t+r} + \sum_{r=0}^{t-2} \binom{s+r-1}{s-1} \frac{(-1)^r}{\alpha^{s+r}(x-\alpha)^{t-r}}}
\]

\[ - (-1)^t \left( \begin{array}{c} s+t-2 \\ s-1 \end{array} \right) \frac{1}{\alpha^{s+t-1}} \left( \frac{1}{x-\alpha} - \frac{1}{x} \right). \]

Now sum over all integers \( 0 < \alpha < x < \infty \).

Nielsen states (10.1) without proof [52, p. 48, eq. (9)]. Markett proves (10.1) by induction [49 Lemma 3.1], which is the proof technique suggested for the \( \alpha = 1 \) case of (10.1) in [8 Lemma 1]. However, it is easy to prove (10.1) directly by expanding the left hand side into partial fractions with the aid of the residue calculus. Alternatively, as in [22] note that (10.1) is an immediate consequence of applying the partial derivative operator

\[
\frac{1}{(s-1)!} \left( - \frac{\partial}{\partial x} \right)^{s-1} \frac{1}{(t-1)!} \left( - \frac{\partial}{\partial y} \right)^{t-1}
\]
to the identity
\[ \frac{1}{xy} = \frac{1}{(x+y)x} + \frac{1}{(x+y)y}, \]
and then setting \( y = \alpha - x \). This latter observation is extended in [22] to establish a \( q \)-analog of another of Euler’s decomposition formulas for \( \zeta(s,t) \).

11. Equating Shuffles and Stuffles

We begin with an informal argument. By the stuffle multiplication rule [10, 14, 18, 19]
\[ \zeta(2)\zeta(1) = \zeta(2,1) + \zeta(1,2) + \zeta(3). \] (11.1)
On the other hand, the shuffle multiplication rule [10, 11, 14, 15, 16] gives \( ab \ast b = 2ab + ba \), whence
\[ \zeta(2)\zeta(1) = 2\zeta(2,1) + \zeta(1,2). \] (11.2)
The identity (11.2) now follows immediately on subtracting (11.1) from (11.2). □

Of course, this argument needs justification, because it involves cancelling divergent series. To make the argument rigorous, we introduce the multiple polylogarithm [10, 13, 14]. For real \( 0 \leq x \leq 1 \) and positive integers \( s_1, \ldots, s_k \) with \( x = s_1 = 1 \) excluded for convergence, define
\[ \zeta_x(s_1, \ldots, s_k) := \sum_{\substack{n_1 > \cdots > n_k > 0}} x^{n_1} \prod_{j=1}^{k} n_j^{-s_j} = \int \prod_{j=1}^{k} \left( \frac{dt_{(j)}}{t_{(j)}^{s_{(j)}}} \right) \frac{dt_{s_j}}{1-t_{(s_j)}}, \] (11.3)
where the integral is over the simplex
\[ x > t_{s_1}^{(1)} > \cdots > t_1^{(k)} > \cdots > t_1^{(s_k)} > 0, \]
and is abbreviated by
\[ \int_0^{x} \prod_{j=1}^{k} a^{s_j-1} b_j, \quad a = \frac{dt}{t}, \quad b = \frac{dt}{1-t}. \] (11.4)

Then
\[ \zeta(2)\zeta_x(1) = \sum_{n>0} \frac{1}{n^2} \sum_{k>0} \frac{x^k}{k} = \sum_{n>k>0} \frac{x^k}{n^2k} + \sum_{k>n>0} \frac{x^k}{k^2} + \sum_{k>0} \frac{x^k}{k^3}, \]
and
\[ \zeta_x(2)\zeta_x(1) = \int_0^{x} ab \int_0^{x} b = \int_0^{x} (2ab + bab) = 2\zeta_x(2,1) + \zeta_x(1,2). \]

Subtracting the two equations gives
\[ [\zeta(2) - \zeta_x(2)] \zeta_x(1) = \zeta_x(3) - \zeta_x(2,1) + \sum_{n>k>0} \frac{x^k - x^n}{n^2k}. \]
We now take the limit as $x \to 1^-$. Uniform convergence implies the right hand side tends to $\zeta(3) - \zeta(2, 1)$. That the left hand side tends to zero follows immediately from the inequalities

$$0 \leq x \left[ \zeta(2) - \zeta_x(2) \right] \zeta_x(1) = x \int_x^1 \log(1 - t) \log(1 - x) \frac{dt}{t} \leq \int_x^1 \log^2(1 - t) \, dt = (1 - x) \left\{ 1 + (1 - \log(1 - x))^2 \right\}.$$ \hfill \square

The alternating case (1.5) is actually easier using this approach, since the role of the divergent sum $\zeta(1)$ is taken over by the conditionally convergent sum $\zeta(1) = -\log 2$. By the stuffle multiplication rule,

$$\zeta(\mathcal{Z})\zeta(\mathcal{T}) = \zeta(\mathcal{Z}, \mathcal{T}) + \zeta(\mathcal{T}, \mathcal{Z}) + \zeta(3), \quad (11.5)$$

$$\zeta(2)\zeta(\mathcal{T}) = \zeta(2, \mathcal{T}) + \zeta(\mathcal{T}, 2) + \zeta(3). \quad (11.6)$$

On the other hand, the shuffle multiplication rule gives $ac \shuffle c = 2ac^2 + cac$ and $ab \shuffle c = abc + acb + cab$, whence

$$\zeta(\mathcal{Z})\zeta(\mathcal{T}) = 2\zeta(\mathcal{Z}, 1) + \zeta(\mathcal{T}, 2), \quad (11.7)$$

$$\zeta(2)\zeta(\mathcal{T}) = \zeta(2, \mathcal{T}) + \zeta(\mathcal{T}, 2) + \zeta(\mathcal{Z}). \quad (11.8)$$

Comparing (11.5) with (11.7) and (11.6) with (11.8) yields the two equations

$$\zeta(\mathcal{Z}, \mathcal{T}) = \zeta(\mathcal{T}, 2) + 2\zeta(\mathcal{Z}, 1) - \zeta(\mathcal{T}, \mathcal{Z}) - \zeta(3),$$

$$\zeta(2, \mathcal{T}) = \zeta(\mathcal{T}, 2) - \zeta(\mathcal{T}, \mathcal{Z}) + \zeta(3).$$

Subtracting the latter two equations yields $2\zeta(\mathcal{Z}, 1) = \zeta(3) + \zeta(3)$, i.e. (2.3), which was shown to be trivially equivalent to (1.5) in §2 \hfill \square

12. Conclusion

There are doubtless other roads to Rome, and as indicated in the introduction we should like to learn of them. We finish with the three open questions we are most desirous of answers to.

- A truly combinatorial proof, perhaps of the form considered in [11].
- A direct proof that the appropriate line integrals in sections 5.8 and 5.9 evaluate to the appropriate multiple of $\zeta(3)$.
- A proof of (1.6), or at least some additional cases of it.
References


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