Finite Euler Products and the Riemann Hypothesis

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Approximations of $\zeta(s)$

A Function Related to $\zeta(s)$ and its Zeros

The Relation Between $\zeta(s)$ and $\zeta_X(s)$
I. Approximations of $\zeta(s)$
The Approximation of $\zeta(s)$ by Dirichlet Polynomials

We write $s = \sigma + it$ and assume $s$ is not near 1.

In the half–plane $\sigma > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$ 

If $X \geq 1$ and we estimate the tail trivially, we obtain

$$\zeta(s) = X \sum_{n=1}^{X} n^{-s} + O\left(\frac{X}{\sigma - 1}\right).$$
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$$\zeta(s) = \sum_{n=1}^{X} n^{-s} + O \left( \frac{X^{1-\sigma}}{\sigma - 1} \right).$$
A crude form of the approximate functional equation extends this into the critical strip:

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s}} = \sum_{n=1}^{X} n^{-s} + O(X^{-\sigma}) \quad (\sigma > 0) \]

But \( X \) must be \( \gg t \).
Approximation by Dirichlet Polynomials in the Strip

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Example

When \( X = t \) we have

\[ \zeta(s) = \sum_{n \leq t} n^{-s} + O(t^{-\sigma}) \quad (\sigma > 0). \]
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**Theorem**

*The Lindelöf Hypothesis is true if and only if*

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\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + O(X^{1/2-\sigma}|t|^\epsilon)
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*for \( \frac{1}{2} \leq \sigma \ll 1 \) and \( 1 \leq X \leq t^2 \).*
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Thus, on LH even short truncations approximate \( \zeta(s) \) well in \( \sigma > 1/2 \).
Approximations when $\sigma \leq \frac{1}{2}$

On the other hand, short sums cannot approximate $\zeta(s)$ well in the strip $0 < \sigma \leq \frac{1}{2}$.

For example, let $\sigma < \frac{1}{2}$ and compare

$$\int_T^t \sum_{n \leq X} n - s \approx T \cdot X^{1 - 2\sigma}$$

and

$$\int_T^t \left| \zeta(\sigma + it) \right|^2 dt \approx T \cdot T^{1 - 2\sigma}.$$

These are not equal if $X$ is small relative to $T$. 
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These are not equal if \( X \) is small relative to \( T \).
The zeta-function also has an Euler product representation

\[ \zeta(s) = \prod_{\rho} \left(1 - \frac{1}{\rho^s}\right)^{-1} \quad (\sigma > 1). \]
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\[ \zeta(s) = \prod_{p \leq X} \left( 1 - \frac{1}{p^s} \right)^{-1} \left( 1 + O\left( \frac{X^{1-\sigma}}{(\sigma - 1) \log X} \right) \right). \]
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Can we extend this into the critical strip?
Yes, but we need to work with a weighted Euler product.

\[\prod_{p \leq X} \left(1 - \frac{1}{p^s}\right)^{-1} = \exp\left(\sum_{p \leq X} \sum_{k=1}^\infty \frac{1}{k} p^{ks}\right) \approx \exp\left(\sum_{n \leq X} \Lambda(n) n^s \log n\right)\]

\[\Lambda(n) = \log p \text{ if } n = p^k, \text{ otherwise } \Lambda(n) = 0.\]
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We "smooth" the \( \Lambda \)'s and call the result \( P_X(s) \).
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Specifically, we set

\[ P_X(s) = \exp \left( \sum_{n \leq X^2} \frac{\Lambda_X(n)}{n^s \log n} \right), \]
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where

$$\Lambda_X(n) = \begin{cases} 
\Lambda(n) & \text{if } n \leq X, \\
\Lambda(n) \left( 2 - \frac{\log n}{\log X} \right) & \text{if } X < n \leq X^2, \\
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\end{cases} \]

Remember

\[ P_X(s) \approx \prod_{p \leq X^2} \left( 1 - \frac{1}{p^s} \right)^{-1}. \]
We also write

\[ Q_X(s) = \exp \left( \sum_{\rho} F_2((s - \rho) \log X) \right) \cdot \exp \left( \sum_{n=1}^{\infty} F_2((s + 2n) \log X) \right) \]

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$$F_2(z) = 2 \int_{2z}^{\infty} \frac{e^{-w}}{w^2} dw - \int_{z}^{\infty} \frac{e^{-w}}{w^2} dw \ (z \neq 0).$$
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For $z$ large $F_2(z)$ is small. For $z$ near 0

$$F_2(z) \sim \log(c z).$$
A Hybrid Formula for $\zeta(s)$

It follows that in the critical strip away from $s = 1$

$$Q_X(s) \approx \prod_{|\rho - s| \leq 1/ \log X} \left( c (s - \rho) \log X \right)$$
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**Theorem (G., Hughes, Keating)**

*For $\sigma \geq 0$ and $X \geq 2$,*

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**Theorem (G., Hughes, Keating)**

For $\sigma \geq 0$ and $X \geq 2$,

$$\zeta(s) = P_X(s) \cdot Q_X(s).$$

Thus, in the critical strip away from $s = 1$

$$\zeta(s) \approx \prod_{\rho \leq X^2} \left( 1 - \frac{1}{\rho^s} \right)^{-1} \cdot \prod_{|\rho - s| \leq 1/\log X} \left( c (s - \rho) \log X \right)$$
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We note that if RH holds and \( \sigma > \frac{1}{2} \), then...
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**Theorem**

Assume RH. Let \( 2 \leq X \leq t^2 \) and \( \frac{1}{2} + \frac{C \log \log t}{\log X} \leq \sigma \leq 1 \) with \( C > 1 \). Then

\[ \zeta(s) = P_X(s) \left(1 + O\left(\log^{(1-C)/2} t\right)\right). \]

Conversely, this implies \( \zeta(s) \) has at most a finite number of complex zeros in this region.
Approximations when $\sigma \leq 1/2$

Short products can *not* approximate $\zeta(s)$ well in the strip $0 < \sigma \leq 1/2$. 

To see this compare, when $\sigma < 1/2$ is fixed and $X < T^{1/2 - \epsilon}$, 

$$\int_{T}^{2T} \left( \log |\zeta(\sigma + it)| \right)^2 dt \sim \left( \frac{1}{2} - \sigma \right)^2 T \log 2 T$$

and

$$\int_{T}^{2T} \left( \log |P_X(\sigma + it)| \right)^2 dt \sim cT \left( X^{2} - 4\sigma \log X \right).$$

If $X$ is a small power of $T$, the second is larger.

The last estimate also shows that if $\sigma < 1/2$, then infinitely often in $t$ 

$$P_X(s) \gg \exp\left( X^{1 - 2\sigma \sqrt{\log X}} \right),$$

which is very large.
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II. A Function Related to $\zeta(s)$ and its Zeros
Deficiency of the Sum Approximation on $\sigma = 1/2$

On LH (and so on RH) we saw that for $\frac{1}{2} < \sigma \leq 1$ fixed,

$$\zeta(s) = \sum_{n \leq X} \frac{1}{n^s} + o(1),$$

even if $X$ is small.
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Here $\chi(s) = \pi^{s-1/2} \Gamma(1/2 - s/2)/\Gamma(s/2)$. 
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So essentially,

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$$
\zeta(s) = \sum_{n\leq X} \frac{1}{n^s} + \chi(s) \sum_{n\leq t/2\pi} \frac{1}{n^{1-s}} + o(1),
$$

we see that

$$
\zeta(\frac{1}{2} + it) = \sum_{n\leq \sqrt{t/2\pi}} \frac{1}{n^{1^2+it}} + \chi(\frac{1}{2} + it) \sum_{n\leq \sqrt{t/2\pi}} \frac{1}{n^{1^2-it}} + o(1).
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Deficiency of the Euler Product Approximation on
\( \sigma = 1/2 \)
How much is the Euler product approximation
\[ \zeta(s) = P_X(s)(1 + o(1)) \]
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But this is far too large if \( X \) is a power of \( t \)
Deficiency of the Euler Product Approximation on $\sigma = 1/2$

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whereas \( \zeta(s) \ll t^\epsilon \).
Definition of $\zeta_X(s)$

As an alternative to

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when $\sigma > 1/2$ is fixed. To study $\zeta_X(s)$ further we need a lemma.

**Lemma**

In $0 \leq \sigma \leq 1$, $|t| \geq 10$, $|\chi(s)| = 1$ if and only if $\sigma = 1/2$.

Furthermore,

$$\chi(s) = \left( \frac{t}{2\pi} \right)^{1/2-\sigma-it} e^{it + i\pi/4} \left( 1 + O(t^{-1}) \right).$$
The Riemann Hypothesis for $\zeta_X(s)$

**Theorem**

*All of the zeros of*

$$\zeta_X(s) = P_X(s) + \chi(s)P_X(\overline{s})$$

*in $0 \leq \sigma \leq 1$ and $|t| \geq 10$ lie on $\sigma = 1/2$.***
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**Proof.**

$$\zeta_X(s) = P_X(s) \left(1 + \chi(s) \frac{P_X(\overline{s})}{P_X(s)}\right).$$

Also, $P_X(s)$ is never 0. Thus, if $s$ is a zero, $|\chi(\sigma + it)| = 1$. By the lemma, when $|t| \geq 10$ this implies that $\sigma = 1/2$. 

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The Number of Zeros of $\zeta_X(s)$

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This vanishes if and only if

$$\frac{1}{2\pi} \arg \chi(1/2 + it) - \frac{1}{\pi} \arg P_X(1/2 + it) \equiv 1/2 \pmod{1}.$$
Detecting Zeros of $\zeta_X(s)$

Set

$$F_X(t) = -\frac{1}{2\pi} \arg \chi(1/2 + it) + \frac{1}{\pi} \arg P_X(1/2 + it).$$
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$$\arg P_X(1/2 + it) = \text{Im} \log P_X(1/2 + it) = \text{Im} \sum_{n \leq X^2} \frac{\Lambda_X(n)}{n^{1/2+it} \log n}$$

$$= - \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(t \log n)}{n^{1/2} \log n}.$$
Lower Bound for the Number of Zeros

So

\[ F_X(t) = \frac{1}{2\pi} t \log \frac{t}{2\pi} - \frac{t}{2\pi} - \frac{1}{8} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(t \log n)}{n^{1/2} \log n} + O\left(\frac{1}{t}\right). \]
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Ignoring the \( O(1/t) \), the condition that \( \zeta_X(1/2 + it) = 0 \) is that this is \( \equiv 1/2 \mod 1 \).
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**Theorem**

\[ N_X(T) \geq \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(T \log n)}{n^{1/2} \log n} + O(1). \]
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How large can the sum be?
Admissible Functions

Call an increasing function $\Phi(t)$ *admissible* if

\[ |S(t)| \leq \Phi(t) \quad \text{and} \quad |\zeta(1/2 + it)| \ll \exp(\Phi(t)). \]
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Montgomery on RH and then Balasubramanian and Ramachandra unconditionally showed that

$$\Phi(t) = \Omega(\sqrt{\log t / \log \log t}).$$
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- $\Phi(t) = (1/2 + \epsilon) \log t / \log \log t$ is admissible on RH.
Conjecture (Farmer, G., Hughes)

\( \Phi(t) = \sqrt{\frac{1}{2} + \epsilon} \log t \log \log t \) is admissible, but
\( \Phi(t) = \sqrt{\frac{1}{2} - \epsilon} \log t \log \log t \) is not.
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Assume RH. Then

\[ \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \sin(t \log n)}{n^{1/2} \log n} \ll \Phi(t) + O \left( \frac{\log t}{\log X} \right). \]
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This is \( \ll \Phi(t) \) if \( X \geq \exp(c \log t/\Phi(t)) \) for some \( c > 0 \).

( Same bound as for \( S(t) \)! )
If $F_X(t)$ is not monotonically increasing, there could be “extra” solutions of

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Now

$$F_X'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \Lambda_X(n) \cos(t \log n) \frac{n^{1/2}}{n^{1/2}} + O\left(\frac{1}{t^2}\right).$$
Extra Solutions

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On RH the sum is \( \ll \Phi(t) \log X. \)
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On RH the sum is $\ll \Phi(t) \log X$.

Thus, on RH there is a positive constant $C$, such that $F_X(t)$ is strictly increasing if

$$X < \exp\left(\frac{C \log t}{\Phi(t)}\right).$$
There are No Extra Solutions When X is Small

We therefore have the

Theorem

Assume RH. There is a constant \( C > 0 \) such that if \( X < \exp(C \log t / \Phi(t)) \), then

\[
N_X(t) = t^2 \pi \log t^2 \pi - t^2 \pi - \frac{1}{\pi} \sum_{n \leq X} \Lambda_X(n) \sin(t \log n) n^{1/2} \log n + O(1).
\]

Unconditionally we can take \( X \) larger, but then we only obtain an asymptotic estimate.

Theorem

If \( X \leq t_0(1) \), then

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N_X(t) \sim t^2 \pi \log t^2 \pi.
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Unconditionally we can take $X$ larger, but then we only obtain an asymptotic estimate.

**Theorem**

*If $X \leq t^{o(1)}$, then*

$$N_X(t) \sim \frac{t}{2\pi} \log \frac{t}{2\pi}.$$
Simple Zeros of $\zeta_X(s)$

$1/2 + i\gamma$ is a simple zero of $\zeta_X(s)$ if $\zeta_X(1/2 + i\gamma) = 0$, but $\zeta'_X(1/2 + i\gamma) \neq 0$. 

This vanishes at $1/2 + i\gamma$ if and only if $F'_X(\gamma) = 0$. 

(University of Rochester)
Simple Zeros of $\zeta_X(s)$

1/2 + i\gamma is a simple zero of $\zeta_X(s)$ if $\zeta_X(1/2 + i\gamma) = 0$, but $\zeta_X'(1/2 + i\gamma) \neq 0$. Now

$$\zeta_X(1/2 + it) = P_X(1/2 + it) \left( 1 + \chi(1/2 + it) \frac{P_X(1/2 - it)}{P_X(1/2 + it)} \right)$$

$$= P_X(1/2 + it) \left( 1 + e^{-2\pi i F_X(t)} \right),$$
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$$= P_X(1/2 + it) \left(1 + e^{-2\pi i F_X(t)}\right),$$

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$$\zeta_X'(1/2 + it) = P_X'(1/2 + it) \left(1 + e^{-2\pi i F_X(t)}\right)$$

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1/2 + iγ is a simple zero of \( \zeta_X(s) \) if \( \zeta_X(1/2 + i\gamma) = 0 \), but \( \zeta'_X(1/2 + i\gamma) \neq 0 \). Now

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\zeta_X(1/2 + it) = P_X(1/2 + it) \left( 1 + \chi(1/2 + it) \frac{P_X(1/2 - it)}{P_X(1/2 + it)} \right)
\]

\[
= P_X(1/2 + it) \left( 1 + e^{-2\pi i F_X(t)} \right),
\]

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\[
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This vanishes at 1/2 + iγ if and only if \( F'_X(\gamma) = 0 \).
The Number of Simple Zeros When $X$ is Small

Recall that if $X$ is not too large,

$$F'_X(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \cos(t \log n)}{n^{1/2}} + O\left(\frac{1}{t^2}\right) > 0.$$
Recall that if \( X \) is not too large,

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F_X'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \frac{\Lambda_X(n) \cos(t \log n)}{n^{1/2}} + O\left(\frac{1}{t^2}\right) > 0.
\]

Thus we have the

Theorem

Assume RH. There is a constant \( C > 0 \) such that if \( X < \exp\left( C \frac{\log t}{\Phi(t)} \right) \), all the zeros of \( \zeta_X(1/2 + it) \) with imaginary part \( \geq 10 \) are simple.

Unconditionally we have

Theorem

If \( X \leq \exp\left( o\left(\log 1 - \epsilon t\right)\right) \), then \( \zeta_X(1/2 + it) \) has \( \sim T/2\pi \log (T/2\pi) \) simple zeros up to height \( T \).
Recall that if $X$ is not too large,

\[ F_X'(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} - \frac{1}{\pi} \sum_{n \leq X^2} \Lambda_X(n) \cos(t \log n) \frac{n^{1/2}}{n^{1/2}} + O\left(\frac{1}{t^2}\right) > 0. \]

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**Theorem**

*Assume RH. There is a constant $C > 0$ such that if $X < \exp\left(C \log t / \Phi(t)\right)$, all the zeros of $\zeta_X(1/2 + it)$ with imaginary part $\geq 10$ are simple.*
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**Theorem**

If $X \leq \exp\left(o(\log^{1-\epsilon} t)\right)$, then $\zeta_X(1/2 + it)$ has $\sim \frac{T}{2\pi} \log \left(\frac{T}{2\pi}\right)$ simple zeros up to height $T$. 
Simple Zeros of $\zeta_X(s)$ When $X$ is Large

A zero $\gamma$ of $\zeta_X(s)$ is simple if and only if $F_X'(\gamma) \neq 0$.

We have just seen that on RH $F_X'(t) > 0$ if $X < \exp(C \log t / \Phi(t))$ (for some $C$), so all zeros are simple.

But even when $X$ is very large, the odds that $F_X'(\gamma) = 0$ are quite small.
A zero $1/2 + i\gamma$ of $\zeta_X(s)$ is simple if and only if $F'_X(\gamma) \neq 0$. 
A zero $1/2 + i\gamma$ of $\zeta_X(s)$ is simple if and only if $F_X'(\gamma) \neq 0$.

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III. The Relation Between $\zeta(s)$ and $\zeta_X(s)$
Comparing $\zeta(s)$ and $\zeta_X(s)$

Here are graphs of $|\zeta(\frac{1}{2} + it)|$ and $|\zeta_X(\frac{1}{2} + it)|$:

Figure: Graphs of $|\zeta(\frac{1}{2} + it)|$ (solid) and $|\zeta_X(\frac{1}{2} + it)|$ (dotted) near $t = 114$ for $X = 10$ and $X = 300$, respectively.
Comparing $\zeta(s)$ and $\zeta_X(s)$

Here are graphs of $2|\zeta(1/2 + it)|$ and $|\zeta_X(1/2 + it)|$: 

![Graphs of $2|\zeta(1/2 + it)|$ and $|\zeta_X(1/2 + it)|$ for $X=10$ and $X=300$, respectively.](image-url)
Comparing $\zeta(s)$ and $\zeta_X(s)$

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graphs}
\caption{Graphs of $2|\zeta(1/2 + it)|$ (solid) and $|\zeta_X(1/2 + it)|$ (dotted) near $t = 114$ for $X = 10$ and $X = 300$, respectively.}
\end{figure}
Figure: Graphs of $2|\zeta(\frac{1}{2} + it)|$ (solid) and $|\zeta_X(\frac{1}{2} + it)|$ (dotted) near $t = 2000$ for $X = 10$ and $X = 300$, respectively.
Comparing $\zeta(s)$ and $\zeta_X(s)$

There are two striking features:

1. Zeros of $\zeta_X\left(\frac{1}{2} + it\right)$ and $\zeta\left(\frac{1}{2} + it\right)$ are close, even for small values of $X$.
2. $|\zeta_X\left(\frac{1}{2} + it\right)|$ seems to approach $2|\zeta\left(\frac{1}{2} + it\right)|$ as $X$ increases.

Why?
Comparing $\zeta(s)$ and $\zeta_X(s)$

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Why?
The Heuristic Reason Why

\[ |\zeta_X(1/2 + it)| \approx 2 |\zeta(1/2 + it)| \]
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\(P_X(s)\) approximates \(\zeta(s)\) in \(\sigma > 1/2\).
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$P_X(s)$ approximates $\zeta(s)$ in $\sigma > 1/2$.

Since $\chi(s)$ is small in $\sigma > 1/2$, $\zeta_X(s) = P_X(s) + \chi(s)P_X(\overline{s})$ also approximates $\zeta(s)$.
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\[ P_X(s) \text{ approximates } \zeta(s) \text{ in } \sigma > 1/2. \]

Since \( \chi(s) \) is small in \( \sigma > 1/2 \), \( \zeta_X(s) = P_X(s) + \chi(s)P_X(\overline{s}) \) also approximates \( \zeta(s) \).

But \( \zeta_X(s) \) approximates \( \mathcal{F}(s) = \zeta(s) + \chi(s)\zeta(\overline{s}) \) even better.
The Heuristic Reason Why
$|ζ_X(1/2 + it)| \approx 2 |ζ(1/2 + it)|$

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Since $χ(s)$ is small in $σ > 1/2$, $ζ_X(s) = P_X(s) + χ(s)P_X(\overline{s})$ also approximates $ζ(s)$.

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On $σ = 1/2$
The Heuristic Reason Why

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On \( \sigma = 1/2 \)
\[ \mathcal{F}(1/2 + it) = \zeta(1/2 + it) + \chi(1/2 + it)\overline{\zeta(1/2 - it)} \]
The Heuristic Reason Why

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But \( \zeta_X(s) \) approximates \( \mathcal{F}(s) = \zeta(s) + \chi(s)\zeta(\overline{s}) \) even better.

On \( \sigma = 1/2 \)

\[
\mathcal{F}(1/2 + it) = \zeta(1/2 + it) + \chi(1/2 + it)\zeta(1/2 - it) \\
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\(P_X(s)\) approximates \(\zeta(s)\) in \(\sigma > 1/2\).

Since \(\chi(s)\) is small in \(\sigma > 1/2\), \(\zeta_X(s) = P_X(s) + \chi(s)P_X(\bar{s})\) also approximates \(\zeta(s)\).

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On \(\sigma = 1/2\)

\[
\mathcal{F}(1/2 + it) = \zeta(1/2 + it) + \chi(1/2 + it)\zeta(1/2 - it)
\]
\[
= \zeta(1/2 + it) + \zeta(1/2 + it)
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But \(\zeta_X(s)\) approximates \(\mathcal{F}(s) = \zeta(s) + \chi(s)\zeta(\overline{s})\) even better.

On \(\sigma = 1/2\)

\[
\mathcal{F}(1/2 + it) = \zeta(1/2 + it) + \chi(1/2 + it)\zeta(1/2 - it) \\
= \zeta(1/2 + it) + \zeta(1/2 + it) \\
= 2\zeta(1/2 + it).
\]

In fact, this suggests that \(\zeta_X(1/2 + it) \approx 2 \zeta(1/2 + it)\).
Why Zeros of $\zeta_X(s)$ and $\zeta(s)$ are Close

$$F_X(t) = -\frac{1}{2\pi} \arg \chi(1/2 + it) + S(t) - \frac{1}{\pi} \text{Im} \sum_{\gamma} F_2(i(t - \gamma) \log X) + E$$
Why Zeros of $\zeta_X(s)$ and $\zeta(s)$ are Close

$F_X(t) = -\frac{1}{2\pi} \arg \chi(1/2 + it) + S(t) - \frac{1}{\pi} \text{Im} \sum_{\gamma} F_2(i(t - \gamma) \log X) + E$

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Zeros of $\zeta_X(1/2 + it)$ occur when $F_X(t) \equiv 1/2 \pmod{1}$. 
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Why Zeros of $\zeta_X(s)$ and $\zeta(s)$ are Close

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$$\ll_I \frac{1}{\log^2 X} \sum_{\gamma} \frac{1}{(t - \gamma)^2} \to 0 \quad \text{as} \quad X \to \infty.$$
Theorem Relating $\zeta_X(s)$ and $\zeta(s)$

A similar argument shows that 

$$\zeta_X\left(\frac{1}{2} + it\right) \rightarrow 2 \zeta\left(\frac{1}{2} + it\right)$$

as $X \rightarrow \infty$, and 

$$\zeta_X\left(\frac{1}{2} + it\right)$$

has no zeros in $I$ for $X$ sufficiently large.
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**Theorem**

Assume RH. Let $I$ be a closed interval between two consecutive zeros of $\zeta(s)$ and let $t \in I$. Then
Theorem Relating $\zeta_X(s)$ and $\zeta(s)$

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Assume RH. Let $\mathcal{I}$ be a closed interval between two consecutive zeros of $\zeta(s)$ and let $t \in \mathcal{I}$. Then

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Later Jon Keating and Eugene Bogomolny used $\zeta_{t/2\pi}(1/2 + it)$ as a heuristic tool for calculating the pair correlation function of the zeros of $\zeta(s)$. 
The general problem is to see what further insights we can gain into the behavior of $\zeta(s)$ and other $L$-functions from these models.

Study the number of zeros of $\zeta_X(s)$ and the number of simple zeros when $X$ is large, say $X = t^{\alpha}$.

$\zeta_X(s)$ approximates $F = \zeta(s) + \chi(s)$ well in $\sigma > 1/2 + \log \log t / \log X$ and on $\sigma = 1/2$ when $X$ is large.

What about in between? Andrew Ledoan is extending these results to the Selberg Class of $L$-functions.
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Another Question

Finite Euler products like \[ \prod_{p \leq X} \left( 1 - \frac{1}{p^s} \right) - 1 \] play a prominent role here and also in the hybrid Euler-Hadamard product representation of \( \zeta(s) \).

Very little is known analytically about the behavior of such products. For instance, how large is \[ \int_0^T \left| \prod_{p \leq X} \left( 1 - \frac{1}{p^s} \right) - 1 \right|^{2k} \, dt \]?

Together with Jon Keating, we are trying to determine the outlines of a theory of such moments, even when \( X \) is much larger than \( T \).

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