Infinitely Many New Quadratic Number Fields
with 2-Class Group of Rank 4
Having Infinite 2-Class Field Tower
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Given an algebraic number field $k$, we denote by $k^1$ the Hilbert 2-class field of $k$, i.e. the maximal abelian unramified extension of $k$ with degree a power of 2. For non-negative integers $n$ we define the Hilbert 2-class field $k^n$ inductively as $k^0 = k$ and $k^{n+1} = (k^n)^1$. Denoting by $C$ the containment symbol, we define $k^0 C k^1 C k^2 C \ldots k^n C \ldots$ to be the 2-class field tower of $k$. We say that the tower is finite if $k^n = k^{n+1}$ for some $n$, with length $n$ if $n$ is minimal, and infinite otherwise.
• It is well known that if \( k \) is an imaginary quadratic number field with the rank of 2-class group \( \mathbb{C}_{k,2} \), i.e. the 2-Sylow subgroup of the ideal class group \( \mathbb{C}_k \) (in the wide sense) of \( k \), greater than or equal to 5, then \( k \) has infinite 2-class field tower.
• It is also well known that for $k$ imaginary quadratic with rank $C_{k,2} = 2$ or 3, then the 2-class field tower of $k$ may be finite or infinite, and that if rank $C_{k,2} = 1$ then the 2-class field tower of $k$ is finite and has length 1.
It was conjectured in the late 1970s that if $k$ is imaginary quadratic with rank $C_{k,2} = 4$, then $k$ has infinite 2-class field tower. From our earlier work (Benjamin, 2001, 2002), in addition to the work of Hajir (1996, 2000), Mouhib (2010), and Sueyoshi (2004, 2009, 2010), we know that if $k$ is as above then $k$ has infinite 2-class field tower in the following cases:
• A) 4-rank of $C_k$ greater than or equal to 3 (Hajir, 1996, 2000)
• B) exactly one negative prime discriminant divides the discriminant $d_k$ of $k$ (Mouhib, 2010)
• C) 4-rank of $C_k$ equal to 2 and five negative prime discriminants divide $d_k$ (Benjamin, 2002)
• D) 4-rank of $C_k$ equal to 2, exactly three negative prime discriminants divide $d_k$, and $d_k$ not congruent to 4 mod 8 (Benjamin, 2002)
• E) 4-rank of $C_k$ equal to 1, five negative prime discriminants divide $d_k$, and $d_k$ not congruent to 4 mod 8 (Sueyoshi, 2009)
• **NOTE:** For the remainder of this talk, unless stated otherwise $k$ will always denote an imaginary quadratic number field with rank $C_{k,2} = 4$. 
In the case when the 4-rank of $C_k$ is equal to 2, exactly three negative prime discriminants divide $d_k$, and $d_k$ is congruent to 4 mod 8, we have shown (Benjamin, 2002) that $k$ has infinite 2-class field tower except for one particular family with certain specified Kronecker symbols of the primes dividing $d_k$, which has been corroborated by Sueyoshi (2004, 2010) using Rédei matrices.
• The above specified Kronecker symbols are as follows, where $p_1$ and $p_2$ are distinct primes dividing positive prime discriminants dividing $d_k$, and $q_1 = 2$ (wlog), $q_2$, and $q_3$ are distinct primes dividing negative prime discriminants dividing $d_k$: $(p_1/q_1) = (p_1/q_2) = (p_1/q_3) = (p_2/q_1) = (p_2/q_2) = (p_2/q_3) = -1$. 
We define “new” imaginary quadratic number fields $k$ with rank $C_{k,2} = 4$ and infinite 2-class field tower, as fields that to the best of our knowledge were not previously known to have infinite 2-class field tower. With this definition of new fields $k$, through a generalization of a result by Mouhib (2010) when $k$ is an imaginary quadratic number field with rank $C_{k,2} = 2$, we have demonstrated that there are new fields $k$ in the above family when the 4-rank of $C_k$ is equal to 2, exactly three negative prime discriminants divide $d_k$, and $d_k$ is congruent to 4 mod 8.
• Through the above generalization of Mouhib’s result, we have also shown that there are new fields $k$ when the 4-rank of $C_k$ is equal to 1, $d_k$ is congruent to 4 mod 8, and either five or exactly three negative prime discriminants divide $d_k$, and infinitely many new fields $k$ when the 4-rank of $C_k$ is equal to 1, exactly three prime discriminants divide $d_k$, and $d_k$ is not congruent to 4 mod 8.
• Our above results all lend support to the 2-class field tower conjecture. To give an indication of how we have obtained our results, we first state the Golod & Shafarevich Inequality (1964) (as refined by Gaschutz and Vinberg—see Koch (1969)), and two related inequalities used by Sueyoshi (2004, 2009, 2010) that have been derived from a more generic inequality by Martinet (1978).
Lemma 1: Golod & Shafarevich Inequality:

- Let $k$ be a number field, $C_k$ be the class group of $k$, and $E_k$ be the group of units of $k$. Then the 2-class field tower of $k$ is infinite if rank $C_{k,2}$ is greater than or equal to $2 + 2(\sqrt{\text{rank}_2(E_k)} + 1)$, where $\text{rank}_2(E_k)$ is the rank of the elementary 2-group $E_k/E_k^2$ (and can be described as the number of infinite primes of $k$).
Lemma 2:

i) Let $F$ be a totally real number field of degree $n$, and $E$ be a totally imaginary quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$ which ramify in $E$. If $t \geq 3 + 2\sqrt{n + 1}$, then the 2-class field tower of $E$ is infinite.
Lemma 2:

• ii) Let $F$ be a totally imaginary number field of degree $n$, and $E$ be a quadratic extension of $F$. Let $t$ be the number of prime ideals of $F$ which ramify in $E$. If $t \geq (n/2) + 3 + 2(\sqrt{n + 1})$, then the 2-class field tower of $E$ is infinite.
• We next state the result of Mouhib (2010) in the rank $C_{k,2} = 2$ case that we have generalized to prove our above results.
Lemma 3:

• Let $k$ be an imaginary quadratic number field with $C_{k,2} = 2$ and 4-rank of $C_k$ equal to 1. Let $p_1$ and $p_2$ be distinct primes numbers such that the class number (in the wide sense) of $\mathbb{Q}(\sqrt{p_1 p_2})$ is divisible by 16. Then for each prime number $q$ congruent to 3 mod 4 such that the Kronecker symbol $(p_1 p_2 / q) = -1$, the 2-class field of $\mathbb{Q}(\sqrt{-p_1 p_2 q})$ is infinite.
• In order to prove this lemma, Mouhib used the following results from genus theory, that we made use of to generalize Mouhib’s above result to our fields $k$ with rank $C_{k,2} = 4$. 
Lemma 4:

• Let $K$ be a quadratic extension of a number field $k$. Then $\text{rank } C_{k,2}$ is greater than or equal to $\text{ram}(K/k) - (\text{dimension over the field } F_2 \text{ of } E_k/(E_k \cap N_{K/k}(K^*)) - 1$, where $\text{ram}(K/k)$ is the number of primes that ramify in the extension $K/k$, $K^*$ is the multiplicative group of $K$, and $N_{K/k}$ is the norm map in the extension $K/k$. Furthermore, the dimension over $F_2$ of $E_k/(E_k \cap N_{K/k}(K^*))$ is less than or equal to $[k:Q]$ if $k$ is totally real, and is less than or equal to $(1/2)[k:Q]$ otherwise, where $Q$ is the field of rational numbers.
Remark 1:

• Recall the well-known result from genus theory that if $k$ is a quadratic number field with discriminant $d_k$, and $t$ is the number of primes that ramify in $k$ (which is the number of primes that divide $d_k$), then $\text{rank } C_{k,2} = t - 2$ if $d_k > 0$ and is not a sum of two squares, and $\text{rank } C_{k,2} = t - 1$ otherwise. To obtain our 4-ranks we utilize the standards technique of $d_k$-splittings of the second kind, as well as an application of Rédei matrices.
We now state our generalization of Mouhib’s above result, where \( h(M) \) refers to the 2-class number of \( M \).
Lemma 5:

• Let $p_i$, $q_i$, $i = 1, 2, 3, 4$, be distinct prime numbers such that $p_i$ is congruent to 1 mod 4 and $q_i$ is congruent to 3 mod 4, and let $q_5$ be a prime number such that $q_5$ is congruent to 3 mod 4, or $q_5 = 2$ if $q_5^* = -4$ or $q_5^* = -8$, and $q_5$ is not equal to $q_i$ for $i = 1, 2, 3$ or 4. Without loss of generality let $M = \mathbb{Q}(\sqrt{p_1 p_2 q_1})$ (resp. $\mathbb{Q}(\sqrt{q_1 q_2 q_3})$, $\mathbb{Q}(\sqrt{q_1 q_2 q_3 q_4})$, $\mathbb{Q}(\sqrt{p_1 p_2 q_1 q_2})$, $\mathbb{Q}(\sqrt{2 q_1 q_2 q_3})$, $\mathbb{Q}(\sqrt{2 p_1 q_1 q_2})$, $\mathbb{Q}(\sqrt{2 p_1 p_2 q_1})$, $\mathbb{Q}(\sqrt{2 p_1 p_2 p_3})$, $\mathbb{Q}(\sqrt{2 p_1 p_2 p_3 p_4})$). Assume that 16 divides $h(M)$, and $(4 p_1 p_2 q_1 / q_5) = -1$ (resp. $(4 q_1 q_2 q_3 / q_5) = -1$, $(q_1 q_2 q_3 q_4 / q_5) = -1$, $(p_1 p_2 q_1 q_2 / q_5) = -1$, $(2 q_1 q_2 q_3 / q_5) = -1$, $(2 p_1 q_1 q_2 / q_5) = -1$, $(2 p_1 p_2 q_1 / q_5) = -1$, $(2 p_1 p_2 p_3 / q_5) = -1$, $(p_1 p_2 p_3 p_4 / q_5) = -1$). Let $L$ be an imaginary quadratic number field with exactly five primes dividing $d_L$, and moreover let $L = \mathbb{Q}(\sqrt{-p_1 p_2 q_1 q_5})$ (resp. $\mathbb{Q}(\sqrt{-q_1 q_2 q_3 q_5})$, $\mathbb{Q}(\sqrt{-q_1 q_2 q_3 q_4 q_5})$, $\mathbb{Q}(\sqrt{-p_1 p_2 q_1 q_2 q_5})$, $\mathbb{Q}(\sqrt{-2 q_1 q_2 q_3 q_5})$, $\mathbb{Q}(\sqrt{-2 p_1 q_1 q_2 q_5})$, $\mathbb{Q}(\sqrt{-2 p_1 p_2 q_1 q_5})$, $\mathbb{Q}(\sqrt{-2 p_1 p_2 p_3 q_5})$, $\mathbb{Q}(\sqrt{-2 p_1 p_2 p_3 p_4 q_5})$). Then $L$ has infinite 2-class field tower.
Proof:

- The proof of Lemma 5 is analogous to the proof of Mouhib’s above rank $C_{k,2} = 2$ result (Lemma 3). We illustrate the proof for the case $M = Q(\sqrt{p_1 p_2 q_1}), L = Q(\sqrt{-p_1 p_2 q_1 q_5})$. From the above remark we see that rank $C_{L,2} = 4$, as $d_L = -4p_1 p_2 q_1 q_5$, and we note that $d_M = 4p_1 p_2 q_1$. We let $F$ be the composite field of $M^1$ and $Q(\sqrt{-q_5})$, and we see that $F$ is a totally complex quadratic extension of the totally real field $M^1$, and $F/L$ is unramified. The prime number $q_5$ is inert in the extension $M/Q$, since $(p_1 p_2 q_1/q_5) = -1$, and therefore the $q_5$-adic place of $M$ is principal and consequently splits completely in $M^1$. Since the number of $q_5$-adic places that ramify in $F/M^1$ is equal to $[M^1:M]$, we obtain $\text{ram}(F/M^1) = [M^1:M] + [M^1:Q] = 3[M^1:M]$. Since $M^1$ is totally real, we know that the dimension of the unit index in Lemma 4, where $k = M^1$ and $K = F$, is less than or equal to $[M^1:Q] = 2[M^1:M]$, and consequently Lemma 4 enables us to conclude that rank $C_{F,2} \geq [M^1:M] - 1 \geq 15$. 
On the other hand, since by the Dirichlet Unit Theorem we know that \( \text{rank}_2(E_F) = [M^1:Q] = 2[M^1:M] \), and we can verify that \( [M^1:M] - 1 \geq 2 + 2\sqrt{2|M^1:M| + 1} \), we see that \( F \) satisfies the Golod & Shafarevich Inequality, as described in Lemma 1. It follows that \( F \) has infinite 2-class field tower, and since \( F/L \) is unramified we are able to conclude that \( L \) also has infinite 2-class field tower, which establishes our lemma for this case. The rest of our cases can be proved in an analogous manner to the case for \( L = Q(\sqrt{-p_1p_2q_1q_5}) \), and we omit the details.
To show there are infinitely many fields with infinite 2-class field tower in certain cases, we can make use of the Chinese Remainder Theorem (CRT) and Dirichlet’s Theorem of Primes in an Arithmetic Progression (DPAP), to establish the following lemma:
Lemma 6:

- Assume there exists a field $k$ that satisfies the conditions of Lemma 5, and let $M$ be the corresponding real quadratic number field given in Lemma 5. Then there exist infinitely many such fields that also satisfy these conditions which have $M$ as its corresponding real quadratic number field, and therefore there exist infinitely many such fields that have infinite 2-class field tower.
Remark 2:

- In the case when \( d_k \) is congruent to 4 mod 8, the 4-rank of \( C_j \) for a corresponding field \( j \) obtained by using the Chinese Remainder Theorem (CRT) and Dirichlet’s Theorem of Primes in an Arithmetic Progression (DPAP) may not be the same as the 4-rank of \( C_k \). To see an example of this, let \( k = \mathbb{Q}(\sqrt{-3.5.7.29}) = \mathbb{Q}(\sqrt{-3045}) \) and \( j = \mathbb{Q}(\sqrt{-3.263.5.7.29}) = (\sqrt{-800835}) \). It can be readily shown that \( C_k \) has 4-rank 1 and \( C_j \) has 4-rank 0, and that the field \( j \) is obtained from the field \( k \) by CRT and DPAP as described in Lemma 6.
• However, if $d_k$ is not congruent to 4 mod 8 then for all fields $j$ in the infinite collection of fields described in Lemma 6, the 4-ranks of $C_k$ and $C_j$ will be the same. It follows that in the case when the 4-rank of $C_k$ is equal to 1, exactly three negative prime discriminants divide $d_k$, and $d_k$ is not congruent to 4 mod 8, once we obtain a new field through Lemma 5 we are able to use Lemma 6 to obtain infinitely many such new fields.
• We state our above results in the form of the following theorem, where once again \( k \) is an imaginary quadratic number field such that rank \( C_{k,2} = 4 \).
Theorem 1:

• There exist new fields $k$ that satisfy the 2-class field tower conjecture when $C_k$ has 4-rank 2, and when $C_k$ has 4-rank 1 with exactly three negative prime discriminants dividing $d_k$ and $d_k$ congruent (resp. not congruent) to 4 mod 8, or when five negative prime discriminants divide $d_k$. 
Furthermore, there are infinitely many new such fields $k$ in the case when the 4-rank of $C_k$ is equal to 1, exactly three negative prime discriminants divide $d_k$, and $d_k$ is not congruent to 4 mod 8.
We now give some examples of our above results, where we utilize Keith Mathews’ number theory website (www.numbertheory.org/php/classnopos.html) to obtain the 2-class numbers of our corresponding real quadratic number fields $M$. 
Example 1:

- \( k = \mathbb{Q}(\sqrt{-5.13.7.827}) = \mathbb{Q}(\sqrt{-376285}) \),
- \( M = \mathbb{Q}(\sqrt{5.13.7.827}) = \mathbb{Q}(\sqrt{376285}) \);

then we have \( h(M) = 16 \), \( (5.13.7.827/2) = -1 \), \( (5/2) = (13/2) = (5/7) = (5/827) = (13/7) = (13/827) = -1 \), \( C_k \) has 4-rank 2, and it can be readily shown from Lemma 5 and our previous results that \( k \) is a new field with infinite 2-class field tower.
Example 2:

• \( k = \mathbb{Q}(\sqrt{-23.19.67.3}) = \mathbb{Q}(\sqrt{-87837}) \);
  \( M = \mathbb{Q}(\sqrt{23.19.67.3}) = \mathbb{Q}(\sqrt{87837}) \); then we have \( h(\mathbb{Q}(\sqrt{23.19.67})) = 16, (-23/3) = 1, (-19/3) = (-67/3) = -1, (23.19.67/3) = -1 \), and we see that \( k \) is a new field with 4-rank of \( C_k \) equal to 1 and five negative prime discriminants dividing \( d_k \).
Example 3:

- $k = \mathbb{Q}(\sqrt{-5.13.7.83}) = \mathbb{Q}(\sqrt{-37765})$, 
  $M = \mathbb{Q}(\sqrt{5.13.7.83}) = \mathbb{Q}(\sqrt{37765})$; then we have $h(M) = 16$, $(5.13.7.83/2) = -1$, and we see that $k$ is a new field with 4-rank of $\text{C}_k$ equal to 1, exactly three negative prime discriminants dividing $d_k$, and $d_k$ congruent to 4 mod 8.
Example 4:

- $k = \mathbb{Q}(\sqrt{-19.11.191.13.41}) = \mathbb{Q}(\sqrt{-21276827})$, 
  $M = \mathbb{Q}(\sqrt{19.11.13.41}) = \mathbb{Q}(\sqrt{111397})$; then we have 
  $h(M) = 16$, 
  $(19/191) = (11/191) = (41/191) = -1$, 
  $(13/191) = 1$, 
  $(19.11.13.41/191) = -1$, and we see 
  that $k$ is a new field with 4-rank of $C_k$ equal to 1, 
  exactly three negative prime discriminants 
  dividing $d_k$, and $d_k$ not congruent to 4 mod 8. 
  From Lemma 6 and Remark 2 we conclude that 
  there are infinitely many new such fields with 
  infinite 2-class field tower.
Remark 3:

• We note that the method we have described above does not work when $C_k$ has 4-rank 0 (i.e. when $C_{k,2}$ is elementary) because in all such cases we obtain that either the corresponding real quadratic number field $M$ in Lemma 5 is of the form $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and therefore we do not have $h(M) \geq 16$, or the kronecker symbol inequality required in Lemma 5 is not satisfied.
Based upon our above results, we conclude with the following open questions, where once again $k$ is an imaginary quadratic number field such that $\text{rank } C_{k,2} = 4$. 
Question 1:

- Do there exist infinitely many new fields $k$ that have infinite 2-class field tower when $C_k$ has 4-rank 1 and five negative prime discriminants divide $d_k$? (from Sueyoshi (2009) we know that $d_k$ must be congruent to 4 mod 8 for such new fields)
Question 2:

• Do there exist infinitely many new fields $k$ that have infinite 2-class field tower when $C_k$ has 4-rank 1 (resp. 4-rank 2), exactly three negative prime discriminants divide $d_k$, and $d_k$ is congruent to 4 mod 8?
Question 3:

• Do there exist new fields $k$ (resp. infinitely many new fields $k$) that have infinite 2-class field tower when $C_k$ has 4-rank 0, for any of the following cases:
• A) $d_k$ congruent to 4 mod 8 and exactly three negative prime discriminants divide $d_k$
• B) $d_k$ not congruent to 4 mod 8 and exactly three negative prime discriminants divide $d_k$
• C) $d_k$ congruent to 4 mod 8 and five negative prime discriminants divide $d_k$
• D) $d_k$ not congruent to 4 mod 8 and five negative prime discriminants divide $d_k$?