Relative Bogomolov Extensions
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The absolute height of algebraic numbers

If \( \alpha \in \overline{\mathbb{Q}} \) has minimal polynomial

\[ f(x) = a_0 x^d + \cdots + a_d = a_0 (x - \alpha_1) \cdots (x - \alpha_d) \in \mathbb{Z}[x], \]

we define

\[ H(\alpha) = \left( a_0 \prod_{i=1}^{d} \max\{1, |\alpha_i|\} \right)^{\frac{1}{d}} = \prod_{v} \max\{1, |\alpha|_v\}. \]

- The last product is over all places of an arbitrary number field \( K \ni \alpha \).
- \( H \) is the “multipliclicative” height; \( h := \log H \) denotes the “logarithmic” or “additive” height.
Properties of $H : \overline{\mathbb{Q}}^\times \to [1, \infty)$

- $H(\alpha) = 1$ iff $\alpha$ is a root of unity, a.k.a. a torsion point of $\mathbb{G}_m(\overline{\mathbb{Q}})$.
- If $\alpha = p/q \in \mathbb{Q}$, $(p, q) = 1$, then $H(\alpha) = \max \{|p|, |q|\}$.
- Galois-invariant (all roots of same irred. poly. have same height)
- If $\lambda \in \mathbb{Q}$, then $H(\alpha^\lambda) = H(\alpha)^{|\lambda|}$ (“scaling”)
- $H(\alpha \beta) \leq H(\alpha)H(\beta)$ (“triangle inequality”)
- Roughly comparable to $\ell^1$- and $\ell^\infty$-norms of coefficients of minimal polynomial, which are easier to compute but don’t play nice.
Other heights

- Absolute height on algebraic numbers \( h : \mathbb{G}_m(\overline{\mathbb{Q}}) \to [0, \infty) \) (the focus for this talk)— more generally on \( \mathbb{G}_m^n \) sum the heights of coordinates.

- Absolute height of a point in projective space \( h : \mathbb{P}^n(\overline{\mathbb{Q}}) \to [0, \infty) \) is just a slight adjustment of the previous definition.

- If \( V/\overline{\mathbb{Q}} \) is a variety and we have a map \( f : V \to \mathbb{P}^n \), this induces a height \( h_f : V(\overline{\mathbb{Q}}) \to [0, \infty) \), by \( h_f(P) = h(f(P)) \).

- For an abelian variety \( A \) we have the Néron-Tate canonical height \( \hat{h} : A(\overline{\mathbb{Q}}) \to [0, \infty) \), which is “close” \( O(1) \) away to \( h_f \) for any \( f \) and respects the group structure and Galois action, analogous to \( h \) on \( \mathbb{G}_m \).

- Fancy results in diophantine geometry (e.g. Faltings’s Theorem) use fancier heights.
Unconditional lower bounds – the Lehmer conjecture

**Conjecture (D. H. Lehmer, ’33)**

There exists an absolute constant $c > 1$ such that if $\alpha$ is an algebraic number of degree $d$ over $\mathbb{Q}$, not a root of unity, then

$$H(\alpha)^d \geq c.$$ 

- Evidence (ask Mike Mossinghoff) suggests that we can take $c = 1.17628...$, achieved when $\alpha$ is a root of $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, already discovered by Lehmer in ’33 (calculating by hand!).

- Dobrowolski (’79) showed we can get $H(\alpha)^d \geq c' \cdot \left(\frac{\log \log d}{\log d}\right)^3$, and Voutier later showed here we can take $c' = 1/4$ for all $d > 1$. 

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Definition (Bombieri & Zannier, '01)

A subfield $K \subseteq \mathbb{Q}$ satisfies the **Bogomolov property** if there exists $\varepsilon > 0$ such that there is no non-torsion point $\alpha \in K^\times$ with $h(\alpha) < \varepsilon$. ("$K$ has no small points.")

- Easy to see all number fields have (B).
- We say $K$ has (B) w.r.t. an abelian variety $A$ if $A(K)$ has no small points in the canonical height.
- Named after the related Bogomolov conjectures in diophantine geometry.
- $K$ has (B) $\iff K^\times / \text{tors}$ is a discrete subgroup of $\mathbb{Q}^\times / \text{tors}$.
- This means that if $K$ has (B), then $K^\times / \text{tors}$ is a free abelian group. Same for $A(K) / \text{tors}$ if $K$ has (B) w.r.t. $A$. 

Fields with (B) – no small points

- $\mathbb{Q}^{\text{tot. real}}$ (Schinzel '73) – $\alpha$ tot. real $\Rightarrow H(\alpha) \geq \frac{1+\sqrt{5}}{2}$ (sharp)

- $\mathbb{Q}^{ab}$ (Amoroso & Dvornicich '00)

If we interpret the above as a result about heights on $\mathbb{G}_m(\mathbb{Q}(\mathbb{G}_m, \text{tors}))$, this generalizes in several ways:

- $\mathbb{G}_m(k^{ab})$, $k$ a number field (Amoroso & Zannier, '00, '10)

- $A(\mathbb{Q}(\mathbb{G}_m, \text{tors}))$, $A/\mathbb{Q}$ an abelian variety (Baker & Silverman '04)

- $\mathbb{G}_m(\mathbb{Q}(E_{\text{tors}}))$ and $E(\mathbb{Q}(E_{\text{tors}}))$, $E/\mathbb{Q}$ an elliptic curve (Habegger '13)

- Any Galois extension of $\mathbb{Q}$ which embeds into a finite extension of $\mathbb{Q}_p$ for some $p$ (i.e. “totaly $p$-adic” (Bombieri & Zannier '01)

- Any extension $L$ of a number field $k$ such that $\text{Gal}(L/k)/\mathbb{Z}(\text{Gal}(L/k))$ has finite exponent (Amroso, David, & Zannier, '13)
The relative Bogomolov property

Definition

Let $K \subseteq L$ be subfields of $\overline{\mathbb{Q}}$. The extension $L/K$ is Bogomolov (or satisfies the relative Bogomolov property, (RB)) if there exists $\epsilon > 0$ such that no non-torsion point $\alpha \in L^\times \setminus K^\times$ has $h(\alpha) < \epsilon$.

(“$L$ has no new small points.”)

- If $K$ has (B), then $L/K$ has (RB) iff $L$ has (B).
- For $M/L/K$, $M/K$ is (RB) iff $M/L$ and $L/K$ are both (RB).
- If $L \setminus K$ has a root of unity and $K$ has small points, so does $L \setminus K$.

Theorem (G., Pottmeyer (indep.))

This can happen even when $K$ has small points.
Examples

1. Let $K = \mathbb{Q}(\sqrt[3]{3}, \sqrt[3]{9}, \sqrt[3]{27}, \ldots)$ (choosing always the real root).
   - $h(3^{1/3n}) = \frac{1}{3^n} \cdot h(3) = \frac{\log 3}{3^n} \to 0$ as $n \to \infty$, so $K$ does not have (B).
   - Note that the only proper subfields of $K$ are the finite extensions $\mathbb{Q}(\sqrt[3]{3})$, $n = 0, 1, 2, \ldots$. Since all proper subfields are finite, $K$ does not have (RB) over any subfield.
   - $K(\sqrt{3})$ does not have (RB) – intuitively, $K$ has points that are “close” to $\sqrt{3}$.
   - $K(\sqrt{p})$ does have (RB) for $p$ any other prime – this is harder to see, but a key fact is that $p$ is unramified in $K$.

2. Let $K = \mathbb{Q}^{\text{tot.real}}(\sqrt{-1})$ (the “maximal CM field”).
   - $K$ has small points (Amoroso & Nuccio '07), even though $\mathbb{Q}^{\text{tot.real}}$ does not (Schinzel).
   - There is no extension $L/K$ having (RB) (Pottmeyer, '13).
Main result

Theorem (G.)

Let $K/\mathbb{Q}$ be a Galois extension. If there is a (finite) rational prime $p$ with bounded ramification index in $K$, then there exist relative Bogomolov extensions $L/K$.

- These extensions are of the form $K(\sqrt[p]{\alpha})$ for an appropriate choice of $\alpha \in K$.
- cf. Bombieri & Zannier: if $K$ has bounded local degree (ram. index times residual degree) at some prime, then $K$ has (B).

Proof ingredients:

- A “non-archimedean” inequality of Silverman ('84) which bounds from below the heights of generators of relative extensions in terms of the relative discriminant (this generalizes a classical bound of Mahler).
- An “archimedean” bound of Garza ('07) which generalizes Schinzel’s theorem for totally real numbers.
It’s over!