Bounds on Height Functions

Justin Sukiennik

Colby College

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Notation

- Let $K$ be a number field.
- $\mathcal{O}_K$ is the ring of integers in $K$.
- $M^0_K$ is the set of non-archimedean absolute values which are extended from $p$-adic absolute values.
- $M^\infty_K$ is the set of archimedean absolute values extended from the usual absolute value.
- $M_K = M^0_K \cup M^\infty_K$ is the set of all absolute values listed above.
The Height Function over a Number Field

Definition

The logarithmic height $h$ over the number field $K$ is a function from the field $K$ to $\mathbb{R}_{\geq 0}$ defined as follows

$$h(x) = \frac{1}{[K : \mathbb{Q}]} \left( \sum_{v \in M_K} n_v \log \max\{1, |x_v|\} \right) \text{ for any } x \in K,$$

where $n_v$ is the local degree $[K_v : \mathbb{Q}_v]$.

When $K = \mathbb{Q}$, the height for $\frac{x}{y} \in \mathbb{Q}$ (in lowest terms) is described by

$$h\left(\frac{x}{y}\right) = \log \max\{|x|, |y|\}.$$
Properties of the Height Function

The definition of the height function $h$ can be extended to $\mathbb{Q}$, i.e. if $x \in K$ and $x \in K'$, then $h_K(x) = h_{K'}(x)$.

We have the following properties for height functions over the algebraic numbers:

1. If $\alpha$ and $\beta \in K$ are conjugates, then $h(\alpha) = h(\beta)$.
2. (Product formula) For any $x \in K^\times$, we have

$$\prod_{v \in M_K} |x|_v^{n_v} = 1,$$

where $n_v$ is the local degree $[K_v : \mathbb{Q}_v]$. 

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Bounds on Height Functions
Important Facts

We have a couple of theorems central to height functions.

**Theorem (Northcott)**

For any $M, N \in \mathbb{R}_{>0}$, there are only finitely many $\alpha \in \overline{\mathbb{Q}}$ such that

$$h(\alpha) \leq M \text{ and } \deg \alpha \leq N.$$ 

**Theorem**

If $\alpha \in \overline{\mathbb{Q}}^\times$, then $h(\alpha) = 0$ if and only if $\alpha$ is a root of unity.

**Example**

$$\left\{ 0, \pm 1, \pm 2, \pm \frac{1}{2}, \pm 3, \pm \frac{1}{3}, \pm \frac{2}{3} \right\}$$

are the only rational $x$ where

$$h(x) \leq \log 3.$$
Canonical Height

Definition
We define the $n^{th}$ iterate of the rational function $\varphi$ as follows
$$\varphi^n(x) = (\varphi \circ \varphi \circ \ldots \circ \varphi)(x). \ (n \text{ times})$$

Definition
The canonical height for $\varphi$ is defined as follows
$$\hat{h}_\varphi(x) = \lim_{n \to \infty} \frac{h(\varphi^n(x))}{d^n}, \text{ where } d = \deg \varphi.$$

Two properties about canonical height to note
1. For $\varphi$ with $\deg \varphi \geq 2$, we have $\hat{h}_\varphi(\alpha) = 0$ if and only if $\alpha$ is a preperiodic point of $\varphi$ or $\alpha = 0$.
2. If $\varphi(x) = x^2$, then $\hat{h}_\varphi = h$, where $h$ is the usual height.
Linear Fractional Transformations

We will focus on linear fractional transformations $\varphi(x) = \frac{ax + b}{cx + d}$ where $a, b, c, d \in \mathcal{O}_K$ have no “common factors” and $ad - bc \neq 0$.

No “common factors” means for all $v \in M^0_K$ there exists $\alpha \in \{a, b, c, d\}$ such that $|\alpha|_v = 1$.

For algebraic numbers $x_1, x_2, \ldots, x_n$, we have three properties:

1. $h(x_1x_2 \cdots x_n) \leq h(x_1) + h(x_2) + \cdots + h(x_n)$,
2. $h(x_1 + x_2 + \cdots + x_n) \leq h(x_1) + h(x_2) + \cdots + h(x_n) + \log n$, and
3. $h(x_1^{-1}) = h(x_1)$ when $x_1 \neq 0$.

Let’s attempt to find an upper bound for $h(\varphi(x)) - h(x)$, for all $x \in K$. 
Now, we can use the naïve bounds on heights to find a bound for the expression

\[ h(\varphi(x)) - h(x) \leq h\left(\frac{ax + b}{cx + d}\right) - h(x), \]

\[ \leq h(ax + b) + h(cx + d) - h(x), \]

\[ \leq h(a) + h(b) + h(c) + h(d) + h(x) + \log 4. \]

Is there a best possible upper bound (not dependent on \( x \in K \)?)
A Theorem about a Strict Upper Bound

**Theorem (S., 2012)**

If \( \varphi(x) = \frac{ax + b}{cx + d} \) as defined previously and \( L \) is the Galois closure of \( K \), then for all \( \beta \in L \), we have

\[
 h(\varphi(\beta)) - h(\beta) \leq \frac{1}{[L : \mathbb{Q}]} \sum_{v \in M_L^\infty} n_v \log \max \{|a|_v + |b|_v, |c|_v + |d|_v\}.
\]

This inequality is the “best possible,” or strict.

**Example**

Define \( \varphi(x) = \frac{-3}{5} x + \frac{4}{3} = \frac{-9x + 20}{15} \). For rational \( \beta \), we have

\[
 h(\varphi(\beta)) - h(\beta) \leq \log \max \{|15|, |-9| + |20|\} = \log 29.
\]
Sketch of Proof

1. First, we get

$$\frac{1}{[K:\mathbb{Q}]} \sum_{v \in M^K_{\infty}} n_v \log \max\{|a|_v + |b|_v, |c|_v + |d|_v\}$$

as an upper bound. (We do not need to pass to the Galois closure for this part.)

2. Then, we need to find $\beta \in L$ that attains or is infinitesimally close to our upper bound.
Theorem (Artin-Whaples approximation theorem)

Let $S = \{v_i : 1 \leq i \leq n\} \subset M_K$ be a finite set of absolute values of $K$. Let $\beta_1, \ldots, \beta_n \in K$. For any $\varepsilon > 0$, there is $\alpha \in K$ such that

$$|\alpha - \beta_i|_{v_i} < \varepsilon,$$

for all $i$.

- Optimally, we would attain the upper bound, but it is difficult or impossible in a number field $K \neq \mathbb{Q}$.
- We use the approximation theorem to find points that approach the bound infinitesimally.
- We need to pass to the Galois closure $L$ in order to proceed.
After using the Product formula, we have

$$h(\varphi(\beta)) = \frac{1}{[L : \mathbb{Q}]} \left( \sum_{\nu \in M_L} n_\nu \log \max \{|a\beta + b|_\nu, |c\beta + d|_\nu\} \right).$$

Our goals are to find $\beta \in L$ with two properties

1. $\log \max \{|a\beta + b|_\nu, |c\beta + d|_\nu\} = \log \max \{1, |\beta|_\nu\}$, for all $\nu \in M^0_L$, and

2. $\log \max \{|a\beta + b|_\nu, |c\beta + d|_\nu\} \\ \approx \log \max \{|a|_\nu + |b|_\nu, |c|_\nu + |d|_\nu\}$, for all $\nu \in M^\infty_L$.

We use the Artin-Whaples approximation theorem to achieve both properties for a particular $\beta \in L$. 
In the archimedean case, we have for any $x \in L$,
\[\sum_{v \in M_{L}^\infty} n_v \log \max \{1, |x|_v\} = \sum_{\iota \in \text{Gal}(L/Q)} \log \max \{1, |\iota(x)|\}.
\]
Assume $\max\{|a|_v + |b|_v, |c|_v + |d|_v\} = |a|_v + |b|_v$.
To maximize the contribution from the archimedean places, $\beta$ must be close to the element $\frac{|\iota(a)|}{|\iota(a)|} \cdot \frac{\iota(b)}{|\iota(b)|}$ on the unit circle.
But chances are $\frac{|\iota(a)|}{|\iota(a)|} \cdot \frac{\iota(b)}{|\iota(b)|}$ is not in $L$.
Because $L$ is the Galois closure of $K$, there exists $\kappa_{\iota} \in L$ such that
\[\left|\frac{|\iota(a)|}{|\iota(a)|} \cdot \frac{\iota(b)}{|\iota(b)|} - \kappa_{\iota}\right| < \epsilon,
\]
where $\iota(x_{\iota}) = \kappa_{\iota}$. 
The last condition $|\beta - x_v|_v < \epsilon$, for all $v \in M_{\infty}$, can be demonstrated as follows.
An Interesting Corollary

Corollary

In general, if $L$ is the Galois closure of $K$, then

$$\sup \{ h(\varphi(\beta)) - h(\beta) : \beta \in L \} \neq \sup \{ h(\beta) - h(\varphi(\beta)) : \beta \in L \}.$$ 

Proof.

Since $\varphi$ is a bijection in $L$, then

$$\sup \{ h(\varphi^{-1}(\beta)) - h(\varphi(\beta)) : \beta \in L \} = \sup \{ h(\varphi^{-1}(\varphi(\beta))) - h(\varphi(\beta)) \} = \sup \{ h(\beta) - h(\varphi(\beta)) \}.$$ 

We can express the inverse as follows $\varphi^{-1}(x) = \frac{dx - b}{-cx + a}$. 
So, by our theorem, we have

\[
\sup \{ h(\beta) - h(\varphi(\beta)) : \beta \in L \} = \frac{1}{[L : Q]} \sum_{v \in M_L^\infty} n_v \log \max \{|a|_v + |c|_v, |b|_v + |d|_v\}.
\]

Example

Over \(\mathbb{Q}\), Let \(\varphi(x) = \frac{-9x + 20}{15}\) and \(\varphi^{-1}(x) = \frac{-15x + 20}{9}\). By the theorem, we get

\[
\sup \{ h(\varphi(\beta)) - h(\beta) : \beta \in \mathbb{Q} \} = \log \max \{|15|, |-9| + |20|\} = \log 29;
\]

\[
\sup \{ h(\beta) - h(\varphi(\beta)) : \beta \in \mathbb{Q} \} = \log \max \{|9|, |-15| + |35|\} = \log 35.
\]
A theorem by C. Petsche, L. Szpiro, and T. Tucker is as follows.

**Theorem**

Let \( \sigma \) be a rational map of degree \( d \geq 2 \) in \( \mathbb{P}^1(\mathbb{C}) \). We have

\[
\lim_{n \to \infty} \frac{1}{d^n} \sum_{\sigma^n(\alpha) = \alpha} h(\alpha) = \lim_{n \to \infty} \frac{1}{2^n} \sum_{\zeta^{2^n} = \zeta} \hat{h}_\sigma(\zeta).
\]

If we take \( \sigma = \varphi^{-1} \circ f \circ \varphi \) where \( f(x) = x^2 \) is the squaring map and \( \varphi(x) = \frac{ax + b}{cx + d} \), then after a few steps we can re-write the equation as

\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{\sigma^n(\alpha) = \alpha} h(\alpha) = \lim_{n \to \infty} \frac{1}{2^n} \sum_{\zeta^{2^n} = \zeta} h(\varphi(\zeta)).
\]