Stable Models and $U_p$ Slope Calculations

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Overview of Talk

Part I

- Slopes of $U_7$ Acting on Modular Forms for $\Gamma_1(49)$

  1. Recall Basic Definitions
  2. State Theorem of Kilford-McMurdy
  3. Explicit Example
  4. Proof Sketch

Part II

- Optimal Models for $X_0(p^2)$ for Slope Calculations

  1. Wish List
  2. A Potentially Useful Family
  3. Some properties and an Example
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Basic Definitions

\( M_k(\Gamma_1(N)), S_k(\Gamma_1(N)) \): classical modular forms and cuspforms
\( M_k(\Gamma_1(N), \epsilon), S_k(\Gamma_1(N), \epsilon) \): subspaces with specified character
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When a prime \( p \) divides \( N \), recall that the Hecke operator, \( U_p \), acts on \( M_k(\Gamma_1(N)) \), preserving these subspaces. The action of \( U_p \) on \( q \)-expansions at infinity is given by

\[
U_p \left( \sum a_n q^n \right) = \sum a_{np} q^n.
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Now, let \( f \) be a normalized eigenform defined over a number field \( K \), so that \( a_p \) is its \( U_p \) eigenvalue. Embed \( K \) into \( \mathbb{C}_p \). Then the \textbf{slope} of \( f \) is the \( p \)-adic valuation of \( a_p \) where \( v(p) = 1 \).
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Note: The slope depends on both \( f \) and the embedding into \( \mathbb{C}_p \).
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When a prime $p$ divides $N$, recall that the Hecke operator, $U_p$, acts on $M_k(\Gamma_1(N))$, preserving these subspaces. The action of $U_p$ on $q$-expansions at infinity is given by

$$U_p \left( \sum a_n q^n \right) = \sum a_{np} q^n.$$

Now, let $f$ be a normalized eigenform defined over a number field $K$, so that $a_p$ is its $U_p$ eigenvalue. Embed $K$ into $\mathbb{C}_p$. Then the **slope** of $f$ is the $p$-adic valuation of $a_p$ where $v(p) = 1$.

**Note:** The slope depends on both $f$ and the embedding into $\mathbb{C}_p$.

**Open Problem:** Determine the slopes of $M_k(\Gamma_1(N), \epsilon)$, as a function of $(p, k, N, \epsilon)$ and the embedding.
Kilford-McMurdy for $\Gamma_1(49)$

Fix a primitive $42^{\text{nd}}$ root of unity, $\zeta$, and let $\chi$ be the Dirichlet character of conductor 49 defined by $\chi(3) = \zeta$. Let $K_1$ and $K_2$ be the 7-adic completions of $\mathbb{Q}[\zeta]$ so that $v(\zeta + 1) > 0$ and $v(\zeta + 4) > 0$ respectively.
Fix a primitive $42$nd root of unity, $\zeta$, and let $\chi$ be the Dirichlet character of conductor $49$ defined by $\chi(3) = \zeta$. Let $K_1$ and $K_2$ be the $7$-adic completions of $\mathbb{Q}[\zeta]$ so that $v(\zeta + 1) > 0$ and $v(\zeta + 4) > 0$ respectively.

(1) $S_k(\Gamma_1(49), \chi^{7k-6})$ is diagonalized by $U_7$ over $K_1$. The slopes of $U_7$ on this space are the values less than $k-1$ in

$$\left\{ \frac{1}{6} \cdot \left\lfloor \frac{9i}{7} \right\rfloor : i \in \mathbb{N} \right\}.$$ 

(2) $S_k(\Gamma_1(49), \chi^{8-7k})$ is diagonalized by $U_7$ over $K_2$. The slopes of $U_7$ on this space are the values less than $k-1$ in

$$\left\{ \frac{1}{6} \cdot \left\lfloor \frac{9i+6}{7} \right\rfloor : i \in \mathbb{N} \right\}.$$ 

(Each slope corresponds to a one dimensional eigenspace.)
Example

Let $\psi(3) = \gamma$ a primitive 21st root of unity. Then $S_2(\Gamma_1(49), \psi)$ has one family defined over $\mathbb{Q}(\gamma, \alpha)$ where $\alpha$ is a root of

$$x^4 + (\gamma^5 + 1)x^3 + (\gamma^{10} - 5\gamma^5 + 1)x^2$$
$$+ (\gamma^{11} - 4\gamma^{10} - \gamma^7 - \gamma^6 - 2\gamma^5 - \gamma^3 + 2\gamma^2 - \gamma)x$$
$$+ (2\gamma^{10} + \gamma^9 + \gamma^8 + \gamma^7 - \gamma^6 - \gamma^5 - \gamma^4 + \gamma^2 + \gamma + 1).$$
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$$a_7 = (\gamma^{11} - \gamma^{10} + \gamma^8 - \gamma^7 - \gamma^6 + \gamma^5 - \gamma^3 + \gamma^2 - 1)\alpha^3$$
$$+ (\gamma^8 - \gamma^6 + \gamma^5 - \gamma^4 - \gamma^3 + \gamma^2)\alpha^2$$
$$+ (4\gamma^{11} - \gamma^6 + \gamma^5 + 4\gamma^4 - \gamma^3 + \gamma^2 - \gamma)\alpha$$
$$- (\gamma^{11} - \gamma^{10} - 3\gamma^9 + \gamma^8 - \gamma^7 - 2\gamma^6 + 2\gamma^5 + \gamma^4 - 3\gamma^3 + 2\gamma^2 + \gamma - 3).$$
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$$-(\gamma^{11} - \gamma^{10} - 3\gamma^9 + \gamma^8 - \gamma^7 - 2\gamma^6 + 2\gamma^5 + \gamma^4 - 3\gamma^3 + 2\gamma^2 + \gamma - 3).$$

The theorem applies over $K_1$ if we take $\gamma = \zeta^8$, since

$$\chi^{7(2)-6} = \chi^8 = \gamma.$$
Choose the uniformizer $\pi_1 = -\zeta^8 + \zeta^6 - \zeta^4 + \zeta$ for $K_1$. Then $\nu(\pi_1) = 1/6$. The roots for $\alpha$ are defined over $K_1$ with the following approximations:

\[
\alpha_1 = 4 + 5\pi_1 + 1\pi_1^2 + 2\pi_1^3 + 3\pi_1^4 + 5\pi_1^5 + \cdots \\
\alpha_2 = 5 + 4\pi_1 + 2\pi_1^2 + 3\pi_1^3 + 4\pi_1^4 + 1\pi_1^5 + \cdots \\
\alpha_3 = 4 + 1\pi_1 + 5\pi_1^2 + 4\pi_1^3 + 1\pi_1^4 + 6\pi_1^5 + \cdots \\
\alpha_4 = 5 + 5\pi_1^2 + 4\pi_1^3 + 4\pi_1^5 + 2\pi_1^6 + \cdots 
\]

Plugging these values into $a_7$ we find $\pi_1$-adic valuations of 1, 2, 3, and 5. So the theorem is verified in this special case.
Outline of the Proof

In order to use $p$-adic analysis to prove results about slopes of classical modular eigenforms for $\Gamma_1(N)$, these are the standard steps:

1. Use the geometry of $X_1(N)$ to embed $M_k(\Gamma_1(N))$ into a natural $p$-adic family of modular forms ("overconvergent" modular forms). Verify that $U_p$ extends to the family.

2. Compute the $U_p$ slopes of all of the overconvergent eigenforms in your family using analytic techniques.

3a. Use a theorem of Coleman to conclude that all of the overconvergent eigenforms of small slope are classical. Thus, you have constructed a certain number of classical eigenforms with specified slopes.

3b. Compare with known formulas for the total number of classical eigenforms with a given character.

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Let \( f : E_1(49) \to X_1(49) \) be the universal generalized elliptic curve over \( X_1(49) \), and \( \omega = f_* \Omega^1_{E_1(49)/X_1(49)} \). Then \( M_k(\Gamma_1(49)) \) is just the holomorphic sections of \( \omega \otimes^k \). Overconvergent forms are holomorphic sections over \( \mathcal{W} = \mathcal{W}_1(49) \), a certain wide-open neighborhood of the cusp \( \infty \), in the analytification of the curve.
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We want to work on \( X_0(49) \), because we have good explicit equations. Fortunately, there are Eisenstein series, \( E_{1,\chi} \) and \( E_{1,\tau} \) which are holomorphic and non-vanishing over \( \mathcal{W} \). Therefore, we can define an isomorphism

\[
M_0(\Gamma_0(49))(D) \cong M_k(\Gamma_1(49), \chi \tau^{k-1})(\mathcal{W}),
\]

where \( D \) is the wide open disk of \( X_0(49) \) over which \( \mathcal{W} \) lies (via the forgetful map).
General Setup - The Picture

\[ W \xrightarrow{D} X_0(49) \]

\[ \tilde{U}_7 \text{ be the induced linear operator on } M_0(\Gamma_0(49))(D). \]
The Explicit Part of the Proof

Now we consider the following explicit model for $X_0(49)$.

$$y^2 - 7xy(x^2 + 5x + 7) - x(x^6 + 7x^5 + 21x^4 + 49x^3 + 147x^2 + 343x + 343) = 0$$

$$z^2 = x(4x^2 + 21x + 28)$$

Here, $x = \eta_1 / \eta_{49}$ and $y = \eta_7^4 / \eta_{49}^4$. Also, $t = x^4 / y$ is a parameter on the genus 0 curve, $X_0(7)$, with divisor $(0) - (\infty)$, which lifts to a parameter on $D$. 
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Taking $s = \sqrt[4]{7}/t$, $M_0(\Gamma_0(49))(D)$ has “basis” $\{s, s^2, s^3, \ldots \}$. Every form has a unique power series expansion in $s$, and the forms of bounded norm are given by $\mathbb{R}_7[[s]] \otimes \mathbb{C}_7$. This implies that the characteristic polynomials of the truncations of the corresponding matrix representing $\tilde{U}_7$ converge in sup norm to the characteristic series of $\tilde{U}_7$. 
A Truncation of the Large Matrix \((k = 1\) shown\)

Write \(\tilde{U}_7(s^i)\) as a power series in \(s\), and put the coefficients in the \(i^{th}\) column. This yields an infinite dimensional matrix that represents \(\tilde{U}_7\) in the basis \(\{s, s^2, \ldots\}\). A truncation of the corresponding matrix of 7-adic valuations, over \(K_1\), is as follows.

\[
\begin{bmatrix}
1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 3/2 \\
1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\
1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2 \\
1/4 & 1/2 & 3/4 & 5/6 & 13/12 & 4/3 & 31/12 \\
1/6 & 5/12 & 1/2 & 3/4 & 1 & 5/4 & 7/3 \\
1/4 & 1/3 & 7/12 & 5/6 & 13/12 & 7/6 & 29/12 \\
1/6 & 5/12 & 2/3 & 11/12 & 1 & 5/4 & 5/2
\end{bmatrix}
\]

Our theorem says that the sequence of slopes should be \(\{1/6, 1/3, 1/2, 5/6, 1, 7/6, 3/2, \ldots\}\) (almost the sequence of column valuations). This will follow if the determinant of each \(j \times j\) truncation is larger than that of any other principle \(j \times j\) minor. To prove that, we consider the “column functions.”
Proposition: Approximations for $\tilde{U}_7(s^i)$ for $1 \leq i \leq 7$ over $K_1(\alpha)$ where $\alpha^4 = -7$ are as follows.

\[
\begin{align*}
\tilde{U}_7(s^1) &\equiv 2\alpha x_1 z/(x(x + x_1^3)), & v_1 = 2, & e_1 \geq 3 \\
\tilde{U}_7(s^2) &\equiv 4\alpha^2 x_1^2 /x, & v_1 = 4, & e_1 \geq 5 \\
\tilde{U}_7(s^3) &\equiv \alpha^3 x^2 + 5\alpha^3 x_1^2 /x, & v_1 = 6, & e_1 \geq 8 \\
\tilde{U}_7(s^4) &\equiv 3\alpha^4 x^2 + 2\alpha^4 x_1^2 (x + 4x_1^3) /x^2, & v_1 = 9, & e_1 \geq 11 \\
\tilde{U}_7(s^5) &\equiv 6\alpha^5 x_1 (x + x_1^3) /x^3, & v_1 = 12, & e_1 \geq 13 \\
\tilde{U}_7(s^6) &\equiv \alpha^6 x_1 (x^2 + 7) /x^3, & v_1 = 14, & e_1 \geq 15 \\
\tilde{U}_7(s^7) &\equiv \alpha^7 /t, & v_1 = 18, & e_1 \geq 19
\end{align*}
\]

(A recursive formula kicks in from there.)

Note: $\frac{1}{12} v_1(f)$ denotes the minimal 7-adic valuation of $f$ over $D$. 
Scaling and reducing the column functions on the stable reduction, we have the following functions and divisors.

\[
\frac{Z}{X(X-1))} = (\infty) + (-1,0) - (0,0) - (1,0)
\]
\[
\frac{1}{X} = 2(\infty) - 2(0,0)
\]
\[
\frac{Z}{X^2} = (1,0) + (-1,0) + (\infty) - 3(0,0)
\]
\[
\frac{(X-1)}{X^2} = 2(1,0) + 2(\infty) - 4(0,0)
\]
\[
\frac{Z(X-1)}{X^3} = 3(1,0) + (-1,0) + (\infty) - 5(0,0)
\]
\[
\frac{(X^2-1)}{X^3} = 2(1,0) + 2(-1,0) + 2(\infty) - 6(0,0)
\]
\[
\frac{Z(X^2-1)}{X^4} = 3(1,0) + 3(-1,0) + (\infty) - 7(0,0).
\]

By Riemann-Roch, no linear combination of the first \(j\) can ever vanish to degree \(j + 1\) at \(\infty\). Thus, the determinant of the \(j^{th}\) truncation approximates the \(j^{th}\) coefficient of the characteristic series and the slopes are as claimed.
Part II - Optimal Models for $X_0(\rho^n)$ for Slope Calculations
Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.
Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.

(1) We must be able to write down a “Banach basis” for the functions on $W_1(p^n)$.
Optimal Models for $X_0(p^n)$ for Slope Calculations

In order to make a similar slope argument more generally, we would need a model with the following properties.

(1) We must be able to write down a “Banach basis” for the functions on $W_1(p^n)$.

**Canonical Example:** Let $W$ be the wide open in $\mathbb{P}^1$ whose $\mathbb{C}_p$-valued points satisfy

$$v(((x - 1)(x - 2)(x - 3)) < 1$$

(the complement of three affinoid disks). Then

$$A_K(W) = K < X, Y, Z > / (XY - p(X - Y), 2XZ - p(X - Z), YZ - p(Y - Z))$$

Think $X = \frac{p}{t-1}$, $Y = \frac{p}{t-2}$ and $Z = \frac{p}{t-3}$ for a parameter $t$ on $\mathbb{P}^1$.

In general, $W_1(p^n)$ is isomorphic to the complement in $\mathbb{P}^1$ of ss affinoid disks (one for each supersingular $j$-invariant).
(2) Parameters should generate the Weierstrass parameters on the “first” supersingular components.

Stable reduction of $X_0(p^3)$ when $p = 12k + 11$ is shown. The left-most genus 0 vertical component is the reduction of $W_1(p^3)$. It intersects the components, $Y^A_{21}$, which have the equation

$$y^2 = x^{(p+1)/i(A)} - 1.$$
Candidate Model for \( X_0(p) \)

\[
t = \left( \frac{\eta_1}{\eta_p} \right)^{e_1} \quad x = \left( \frac{\frac{dt}{t}}{(\eta_1 \eta_p)^2} \right)^{e_2}
\]

If \( p = 12k + 1 \), we have: \((e_1, e_2) = (2, 6)\) and

\[
(t) = k(0) - k(\infty)
\]

\[
(x)_{\text{neg}} = -(6k + 1)(0) - (6k + 1)(\infty).
\]

If \( p = 12k + 5 \), we have \((e_1, e_2) = (6, 2)\) and

\[
(t) = (3k + 1)(0) - (3k + 1)(\infty)
\]

\[
(x)_{\text{neg}} = -(2k + 1)(0) - (2k + 1)(\infty).
\]

If \( p = 12k + 7 \), we have \((e_1, e_2) = (4, 3)\) and

\[
(t) = (2k + 1)(0) - (2k + 1)(\infty)
\]

\[
(x)_{\text{neg}} = -(3k + 2)(0) - (3k + 2)(\infty).
\]

If \( p = 12k + 11 \), we have \((e_1, e_2) = (12, 1)\) and

\[
(t) = (6k + 5)(0) - (6k + 5)(\infty)
\]

\[
(x)_{\text{neg}} = -(k + 1)(0) - (k + 1)(\infty).
\]
Important Fact: The Atkin-Lehner involution, $w_1$, fixes $x$ and satisfies

$$w_1^* t = \frac{p(e_1/2)}{t}.$$ 

Example: $X_0(17)$ has the equation:

$$t^3 x^4 + (-3934t^3)x^3 + (-8608t^4 + 2667641t^3 - 42291104t^2)x^2$$
$$+ (-2944t^5 - 408968t^4 - 38771644t^3 - 2009259784t^2 - 71061003136t)x$$
$$- 256t^6 - 79328t^5 - 11950529t^4 - 1059834654t^3$$
$$- 58712948977t^2 - 1914785073632t - 30358496383232 = 0$$

It’s actually much nicer. For example, $f(0, t) = t^3 \cdot [g(t) + g(\frac{17^3}{t})]$, where

$$g(t) = -256t^3 - 79328t^2 - 11950529t - 529917327.$$

It’s almost certainly possible to compute slopes for specific $p$ using this model - less clear what can be done in general.