Rational Points and Hypergeometric Functions

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The Goal

Motivation- Igusa’s result

Background
   Definitions
   Koblitz’s Theorem
   The Gross-Koblitz Formula

The main result

A familiar example

Work in progress
Let $X_\lambda$ be the family of varieties defined by

$$X_\lambda : x_1^d + \cdots + x_n^d - d\lambda x_1^{h_1} \cdots x_n^{h_n} = 0$$

where each $h_i$ is a positive integer, $\sum h_i = d$ and $\gcd(d, h_1, \ldots, h_n) = 1$ and $\lambda \in \mathbb{F}_q$.

Let $N_{\mathbb{F}_q}(\lambda)$ be the number of $\mathbb{F}_q$-points on $X_\lambda$. 

Objective: Find an explicit relation between the function $N_{\mathbb{F}_q}(\lambda)$ and hypergeometric functions.
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Motivation - Igusa's result

It is known that for the Legendre family of elliptic curves:

\[ E_\lambda : y^2 = x(x - 1)(x - \lambda), \]

we get that

\[ N_{\mathbb{F}_p}(\lambda) \equiv (-1)^{\frac{p-1}{2}} \left[ 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1 \big| \lambda \right) \right]^{\frac{p-1}{2}}_0 \mod p. \]

We also know that \( 2F_1(\frac{1}{2}, \frac{1}{2}; 1|\lambda) \) is the only holomorphic solution around 0 of the Picard-Fuchs differential equation satisfied by the periods of \( E_\lambda \).
The generalized hypergeometric function

Let $A, B \in \mathbb{N}$. A **hypergeometric function** is a function on $\mathbb{C}$ of the form:

$$A F_B(\alpha; \beta|z) = A F_B(\alpha_1, \ldots, \alpha_A; \beta_1, \ldots, \beta_B|z)$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_A)_k}{(\beta_1)_k \cdots (\beta_B)_k k!} z^k,$$

where $\alpha \in \mathbb{Q}^A$ are **numerator parameters**, $\beta \in \mathbb{Q}^B$ are **denominator parameters**, and the Pochhammer notation is defined by:

$$(x)_k = x(x + 1) \cdots (x + k - 1) = \frac{\Gamma(x + k)}{\Gamma(x)}.$$
Let $\chi_{1/(q-1)} : \mathbb{F}_q^* \to K^*$ be a fixed generator of the character group of $\mathbb{F}_q^*$ where $K$ is $\mathbb{C}$ or $\mathbb{C}_p$. 
Gauss sums

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- For $s \in \frac{1}{q-1} \mathbb{Z}/\mathbb{Z}$ we let $\chi_s = (\chi_{1/(q-1)})^{s(q-1)}$, and for any $s$ set $\chi_s(0) = 0$. 
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- Let $\psi : \mathbb{F}_q \rightarrow K^*$ be a (fixed) additive character.
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- Let $\psi : \mathbb{F}_q \rightarrow K^*$ be a (fixed) additive character.
- For $s \in \frac{1}{(q-1)}\mathbb{Z}/\mathbb{Z}$ we let $g(s)$ denote the Gauss sum
  $$g(s) = \sum_{x \in \mathbb{F}_q} \chi_s(x)\psi(x)$$
A large group action

Let

\[ X_\lambda : \sum_{i=1}^{n} x_i^d - d\lambda x_1^{h_1} \cdots x_n^{h_n} = 0 \]

where each \( h_i \) is a positive integer, \( \sum h_i = d \) and \( \gcd(d, h_1, \ldots, h_n) = 1 \).
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- Let $\mu_d^n$ be the group of $n$-tuples of $d$-th roots of unity in $\mathbb{F}_q^*$.  
- Let $\Delta$ be the diagonal elements of $\mu_d^n$, i.e. elements of the form $(\xi, \ldots, \xi)$. 

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The varieties \( X_\lambda \) allow a faithful action of the group

\[ G = \{ \xi \in \mu_d^n | \xi^h = 1 \}/\Delta, \]

by \( \xi = (\xi_1, \ldots, \xi_n) \) taking the point \((x_1, \ldots, x_n)\) to \((\xi_1x_1, \ldots, \xi_nx_n)\).
A large group action

\[ \text{char}(G) \leftrightarrow W, \]

where

\[ W = \{(w_1, \ldots, w_n) | 0 \leq w_i < d, \sum w_i \equiv 0 \pmod{d}\}, \]

and \( w' \sim w \) if \( w - w' \) is a multiple (mod d) of \( h \).

Here

\[ \chi_w(\xi) := \chi(\xi^w), \quad \xi^w = \xi_1^{w_1} \cdots \xi_n^{w_n} \]

and \( \chi \) is a fixed primitive character of \( \mu_d \), which we can get for example by restricting \( \chi_1/(q-1) \) to \( \mu_d \).
Assume $d | q - 1$.

**Theorem (Koblitz)**

\[
N_{\mathbb{F}_q}(\lambda) = N_{\mathbb{F}_q}(0) + \frac{1}{q - 1} \sum_{\substack{s \in \frac{d}{q-1} \mathbb{Z}/\mathbb{Z} \atop w \in W}} \frac{g \left( \frac{w + sh}{d} \right)}{g(s)} \chi_s(d \lambda),
\]

where we denote \( g \left( \frac{w + sh}{d} \right) = \prod_i g \left( \frac{w_i + sh_i}{d} \right) \).
The Gross-Koblitz formula

Fix our attention on $\mathbb{F}_p$-points on our varieties.
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Fix our attention on $\mathbb{F}_p$-points on our varieties.
We want to find an explicit relation between $N_{\mathbb{F}_p}(\lambda) \mod p$ and generalized hypergeometric functions. We use

\[ g(s) = -\left(\frac{-p}{\Gamma_p(s)}\right). \]

Here, $\Gamma_p$ is the $p$-adic analog of the Gamma function.
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**Theorem (Gross-Koblitz)**

\[ g(s) = \frac{1}{s} \Gamma_p(s). \]

Here, $\Gamma_p$ is the $p$-adic analog of the Gamma function.
The 0-dimensional family

Study $N_{\mathbb{F}_p}(\lambda) \mod p$ for the family

$$Z_\lambda : x_1^d + x_2^d - d\lambda x_1 x_2^{d-1} = 0.$$  

Assume $p$ is a prime such that $d|p - 1$. 

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Z_\lambda : x_1^d + x_2^d - d\lambda x_1 x_2^{d-1} = 0.
\]

Assume \( p \) is a prime such that \( d \mid p - 1 \). We use the following:

**Formula (S)**

\[
N_{\mathbb{F}_p}(\lambda) = N_{\mathbb{F}_p}(0) + \frac{1}{p - 1} \sum_{a=0}^{p-2} (-p)^{\eta(a)} \Gamma_p \left( \frac{a}{p-1} \right) \Gamma_p \left( \left\{ \frac{(d-1)a}{p-1} \right\} \right) \Gamma_p \left( \left\{ \frac{da}{p-1} \right\} \right) \omega(d\lambda)^{-da}
\]

where \( \eta(a) = \left( -\frac{a}{p-1} + \left\{ \frac{(d-1)a}{p-1} \right\} - \left\{ \frac{da}{p-1} \right\} \right) \).

**Notation**

- \( \omega : \mathbb{F}_p^* \to \mathbb{C}_p^* \) - Teichmüller character. (\( \omega(x) \equiv x \mod p \))
- \( \{x\} = x - [x] \), fractional part of \( x \).
The 0-dimensional family

Theorem (S)

Let $\alpha^{(0)} = \left( \frac{1}{d}, \ldots, \frac{d-1}{d} \right)$, $\beta^{(0)} = \left( \frac{1}{d-1}, \ldots, \frac{d-2}{d-1} \right)$.

\[
N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) \\
\equiv \sum_{i=0}^{d-2} \left[ dF_{d-1}(\alpha^{(i)}; \beta^{(i)}|(d-1)^{-1}(d-1)\lambda^{-1}) \right] \frac{(i+1)(p-1)}{d} \frac{(p-1)}{d-1} - 1 \mod p,
\]

where $\alpha^{(i)} = \left( \frac{1}{d} + 1, \ldots, \frac{i}{d} + 1, \frac{i+1}{d}, \ldots, \frac{d-1}{d} \right)$, and $\beta^{(i)} = \left( \frac{1}{d-1} + 1, \ldots, \frac{i}{d-1} + 1, \frac{i+1}{d-1}, \ldots, \frac{d-2}{d-1} \right)$.

$[u(z)]_i^j$ denotes the polynomial which is the truncation of a series $u(z)$ from $n = i$ to $j$. 
The 0-dimensional family

So for example in the case $d = 3$ we get that

$$N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) \equiv \left[ 2F_1 \left( \frac{1}{3}, \frac{2}{3}; \frac{1}{2} \bar{x^3} \right) \right]_0^{\frac{p-1}{3}-1}$$

$$+ \left[ 2F_1 \left( \frac{4}{3}, \frac{2}{3}; \frac{3}{2} \bar{x^3} \right) \right]^{\frac{2(p-1)}{3}-1}_{\frac{p-1}{2}} \mod p.$$
The Dwork family with \( d = 4 \)

\[
Y_\lambda : x_1^4 + x_2^4 + x_3^4 + x_4^4 - 4\lambda x_1 x_2 x_3 x_4 = 0.
\]

The set \( W \) is made up of 64 vectors, but we can split them up into 16 equivalence classes, and of those there are only three “types”. These are

\[
(0, 0, 0, 0), (1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3)
\]
\[
(0, 1, 1, 2), (1, 2, 2, 3), (2, 3, 3, 0), (3, 0, 0, 1)
\]
\[
(0, 0, 2, 2), (1, 1, 3, 3), (2, 2, 0, 0), (3, 3, 1, 1)
\]

The rest are permutations of these. So there is one class of the first type, 12 classes of the second type, and 3 classes of the third type.
The Dwork family with $d = 4$

\[
N_{\mathbb{F}_p}(\lambda) - N_{\mathbb{F}_p}(0) = \frac{1}{p-1} \sum_{s \in \frac{1}{p-1} \mathbb{Z}/\mathbb{Z}} \frac{g(s)^4}{g(4s)} \chi_{4s}(4\lambda) \quad (S_1)
\]

\[
+ \frac{12}{p-1} \sum_{s \in \frac{1}{p-1} \mathbb{Z}/\mathbb{Z}} \frac{g(s)g(s + \frac{1}{4})^2g(s + \frac{1}{2})}{g(4s)} \chi_{4s}(4\lambda) \quad (S_2)
\]

\[
+ \frac{3}{p-1} \sum_{s \in \frac{1}{p-1} \mathbb{Z}/\mathbb{Z}} \frac{g(s)^2g(s + \frac{1}{2})^2}{g(4s)} \chi_{4s}(4\lambda). \quad (S_3)
\]
The Dwork family with $d = 4$

Using Gross-Koblitz and taking mod $p$ leaves only $(S_1)$, so

$$N_{IF_p}(\lambda) - N_{IF_p}(0) \equiv \left[ \genfrac{3}{0}{}{3}{2} \left( \frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, 1 \middle| \lambda^{-4} \right) \right]_{0}^{\frac{p-1}{4}} \mod p$$
What’s Next

- Extend the results for $\mathbb{F}_p$ points to $\mathbb{F}_q$. 
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- Prove a similar result for general families of the form $X_\lambda$.
- Relate the number of points to eigenvalues of Frobenius.
Thanks for the invitation!