Problems for Seniors

1. Consider a parallelepiped with the edges a, b, and c. If $b = 2a$, $c = 3a$, and all three angles between a and b, between b and c, and between a and c are 60^o, find the angle between the longest and the shortest diagonals of the parallelepiped.

2. Find all solutions of the differential equation $y' = y - e^{-x^2}$ such that $y \to 0$ as $x \to -\infty$ and as $x \to \infty$.

3. Prove that $\int_0^\infty \frac{dx}{(1+x^2)(1+x^a)}$ does not depend on a.

4. Suppose polynomials $p(x)$ and $q(x)$ have integer values for exactly the same values of x. Prove that either $p(x) - q(x)$ or $p(x) + q(x)$ is a constant.

5. Prove that a hexagon $ABCA'B'C'$ with parallel opposite sides can be inscribed in a circle if and only if its diagonals AA', BB' and CC' are equal.

6. Suppose the distance between points A and B in space is 1. If a point C is selected randomly on a distance less than 1 from A, what is the probability that C is twice closer to A than to B?

7. Determine, with proof, the coefficients of x^{2007} and x^{2008} of the Maclaurin expansion of $[(1+x)(1+x^2)(1+x^4)(1+x^8)]^{-1}$.

8. Let a, b be positive integers such that neither a nor b is divisible by 3 but $a + b$ is divisible by 3. Prove that $x^2 + x + 1$ is a factor of $x^a + x^b + 1$.

9. Prove (without using Fermat's Last Theorem) that if x, y, z are natural numbers, and $n \geq z$, then the relation $x^n + y^n = z^n$ cannot hold.

10. Evaluate the n-fold integral $\int\int ... \int dx_1 dx_2 ... dx_n$ over the region $x_1 \geq 0$, $x_2 \geq 0$,..., $x_n \geq 0$, $x_1 + x_x + ... + x_n \leq 1$.