

### Problems for Sophomores and Juniors

1. Give an example of a differentiable function  $g$  with the following property, or prove that such an example is not possible:
  - (a)  $g'$  has a horizontal asymptote, but  $g$  doesn't have one;
  - (b)  $g$  has a horizontal asymptote, but  $g'$  doesn't have one.
2. How does the number of real roots of the equation  $e^x = ax^3$  depend on the parameter  $a$ ?
3. Prove that the equation  $\sum_{k=0}^{2n} \frac{x^k}{k!} = 0$  has no real roots.
4. Find a function  $f(x)$  on  $[0, \infty)$  such that a substitution  $y = \int_0^x f(t)dt$  transforms  $f(x)$  to  $e^{-y}$ .
5. Prove that  $\int_0^{\infty} \frac{dx}{(1+x^2)(1+x^a)}$  does not depend on  $a$ .
6. Prove that a hexagon  $ABCA'B'C'$  with parallel opposite sides can be inscribed in a circle if and only if its diagonals  $AA'$ ,  $BB'$  and  $CC'$  are equal.
7. Suppose the distance between points A and B in a plane is 1. If a point C is selected randomly on a distance less than 1 from A, what is the probability that C is twice closer to A than to B?
8. Determine, with proof, the coefficients of  $x^{2007}$  and  $x^{2008}$  of the Maclaurin expansion of  $[(1+x)(1+x^2)(1+x^4)(1+x^8)]^{-1}$ .
9. Prove that  $x^9 - 6x^7 + 9x^5 - 4x^3$  is divisible by 8640 for all integer values of  $x$ .
10. Prove (without using Fermat's Last Theorem) that if  $x, y, z$  are natural numbers, and  $n \geq z$ , then the relation  $x^n + y^n = z^n$  cannot hold.